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The Bloch–Okounkov theorem for congruence subgroups and Taylor coefficients of quasi-Jacobi forms

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Abstract

There are many families of functions on partitions, such as the shifted symmetric functions, for which the corresponding q -brackets are quasimodular forms. We extend these families so that the corresponding q -brackets are quasimodular for a congruence subgroup. Moreover, we find subspaces of these families for which the q -bracket is a modular form. These results follow from the properties of Taylor coefficients of strictly meromorphic quasi-Jacobi forms around rational lattice points.

Keywords: Partitions, Jacobi forms, Modular forms; Quasimodular

1 Introduction

Denote by \mathcal{P} the set of all partitions of integers. For a large class of functions $f: \mathcal{P} \rightarrow \mathbb{C}$ the q -bracket, which is defined as the formal power series

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} \in \mathbb{C}[[q]],$$

is a quasimodular form for $SL_2(\mathbb{Z})$ (here λ is a partition of size $|\lambda|$). It is, therefore, natural to raise the following two questions:

- (I) Given a congruence subgroup $\Gamma \leq SL_2(\mathbb{Z})$, is there an (even larger) class of functions for which the q -bracket is a *quasimodular form* for Γ ?
- (II) What is the class of functions for which the q -bracket is a *modular form* (and not just a quasimodular form)?

In this paper, we explain how to answer these questions by studying a different question of independent interest:

- (III) What is the modular or quasimodular behaviour of the Taylor coefficients of meromorphic quasi-Jacobi forms?

To illustrate how these questions are related, consider the *shifted symmetric functions* $Q_k : \mathcal{P} \rightarrow \mathbb{Q}$, for $k \geq 1$ given by

$$Q_k(\lambda) := \beta_k + \frac{1}{(k-1)!} \sum_{i=1}^{\infty} ((\lambda_i - i + \frac{1}{2})^{k-1} - (-i + \frac{1}{2})^{k-1}), \tag{1}$$

where $\sum_{k=0}^{\infty} \beta_k z^{k-1} := \frac{e^{z/2}}{e^z - 1}$. By the celebrated Bloch–Okounkov theorem, for every homogeneous polynomial in these functions, the q -bracket is a quasimodular form for $SL_2(\mathbb{Z})$.

More precisely, we have the following. Here, and throughout the paper, we let $\tau \in \mathfrak{h}$, the complex upper half plane, $z \in \mathbb{C}$ and write $e(x) := e^{2\pi i x}$, $q = e(\tau)$. Given $\lambda \in \mathcal{P}$, write

$$W_\lambda(z) := \sum_{i=1}^{\infty} e((\lambda_i - i + \frac{1}{2})z) \tag{2}$$

for the generating series of the functions Q_k [1, 25], which converges for $\text{Im}(z) < 0$. Write $W(z_1) \cdots W(z_n)$ for the function $\lambda \mapsto W_\lambda(z_1) \cdots W_\lambda(z_n)$. Then, by the Bloch–Okounkov theorem, $\langle W(z_1) \cdots W(z_n) \rangle_q$ is a meromorphic quasi-Jacobi form F_n , defined as follows.

Definition 1.1 For all $n \geq 1$, let \mathfrak{S}_n be the symmetric group on n letters and

$$\theta(\tau, z) := \sum_{v \in \mathbb{F}} (-1)^{\lfloor v \rfloor} e(vz) q^{v^2/2} \quad (\mathbb{F} = \mathbb{Z} + \frac{1}{2})$$

the *Jacobi theta series*. Define the *Bloch–Okounkov n -point functions* F_n by

$$F_n(\tau, z_1, \dots, z_n) := \sum_{\sigma \in \mathfrak{S}_n} V_n(\tau, z_{\sigma(1)}, \dots, z_{\sigma(n)}),$$

where the functions V_n are defined recursively by $V_0(\tau) = 1$ and

$$\sum_{m=0}^n \frac{(-1)^{n-m}}{(n-m)!} \theta^{(n-m)}(\tau, z_1 + \dots + z_m) V_m(\tau, z_1, \dots, z_m) = 0. \tag{3}$$

Here, $\theta^{(r)}(\tau, z) = \left(\frac{1}{2\pi i} \frac{\partial}{\partial z}\right)^r \theta(\tau, z)$ for $r \geq 0$.

The study of the Taylor coefficients of F_n around rational values of z_1, \dots, z_n yields an answer to Question I, whereas a detailed description of the (quasi)modular transformation of the Taylor coefficients of F_n around $z_i = 0$ answers Question II. The Fourier coefficients of the functions F_n were studied in [5], and the Taylor coefficients of certain functions closely related to holomorphic quasi-Jacobi forms were studied in [4]. However, the *Taylor coefficients* of F_n or of a more general meromorphic quasi-Jacobi form have not been studied before.

We now give a short overview of the results in the literature as well as in the present paper. It should be noted that the answer to the third question can be read independently from the applications to functions on partitions.

1.1 The Bloch–Okounkov theorem for congruence subgroups

Given $a \in \mathbb{Q}$, define $Q_k(\cdot, a): \mathcal{P} \rightarrow \overline{\mathbb{Q}}$ by $Q_0(\lambda, a) := \beta_0(a)$ and for $k \geq 1$ by

$$Q_k(\lambda, a) := \beta_k(a) + \frac{1}{(k-1)!} \sum_{i=1}^{\infty} (e(a)^{\lambda_i - i} (\lambda_i - i + \frac{1}{2})^{k-1} - e(a)^{-i} (-i + \frac{1}{2})^{k-1}), \tag{4}$$

where $\sum_{k \in \mathbb{Z}} \beta_k(a) (2\pi iz)^{k-1} := \frac{e(z/2)}{e(z+a)-1}$. We recall $e(x) = e^{2\pi ix}$. The main properties satisfied by these functions are a consequence of the fact that

$$\sum_{k \geq 0} Q_k(\lambda, a) z^{k-1} = e(-\frac{1}{2}a) W_\lambda(z+a),$$

and that $F_n(z_1, \dots, z_n) = \langle W(z_1) \cdots W(z_n) \rangle_q$ is a quasi-Jacobi form. Observe that $Q_k(\lambda, 0) = Q_k(\lambda)$. Up to a constant, these functions $Q_k(\cdot, a)$ have been considered before in [11] for $a = \frac{1}{2}$ and in [10] for all $a \in \mathbb{Q}$. It was shown that a suitably adapted q -bracket of any polynomial in these functions, excluding the function $Q_1(\cdot, a)$, is quasimodular for $\Gamma_1(N)$ for some N .

In this work, we will not change the q -bracket, nor exclude any of the functions (4), and nevertheless prove the following result for the graded algebra $\Lambda^*(N)$, contained in $\mathbb{C}^{\mathcal{P}}$, given by

$$\Lambda^*(N) := \mathbb{Q} \left[Q_k(\cdot, a) \mid k \geq 1, a \in \{0, \frac{1}{N}, \dots, \frac{N-1}{N}\} \right], \tag{5}$$

with the grading given by assigning weight k to $Q_k(\cdot, a)$. For example, a basis for the elements of $\Lambda^*(2)$ of (homogeneous) weight 3 is given by

$$\{Q_3, Q_3(\cdot, \frac{1}{2}), Q_2 Q_1, Q_2 Q_1(\cdot, \frac{1}{2}), Q_2(\cdot, \frac{1}{2}) Q_1, Q_2(\cdot, \frac{1}{2}) Q_1(\cdot, \frac{1}{2}), Q_1^3, Q_1^2 Q_1(\cdot, \frac{1}{2}), Q_1 Q_1(\cdot, \frac{1}{2})^2, Q_1(\cdot, \frac{1}{2})^3\}.$$

Observe that there are no elements of negative weight in $\Lambda^*(N)$.

Given $N \geq 1$, write $(2, N)$ for $\gcd(2, N)$. For $\widehat{N} \in \mathbb{Z}$, denote $m_{\widehat{N}} = \begin{pmatrix} \widehat{N} & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 1.2 *Let $k \in \mathbb{Z}, N \geq 1$ and $\widehat{N} = (2, N)N$. For $f \in \Lambda^*(N)$ of weight k , the q -bracket $\langle f \rangle_q$ is a quasimodular form of weight k for $m_{\widehat{N}}^{-1} \Gamma(\widehat{N}) m_{\widehat{N}}$.*

Remark The occurrence of the group $m_{\widehat{N}}^{-1} \Gamma(\widehat{N}) m_{\widehat{N}}$ in the theorem can be interpreted in at least two ways. First of all, an equivalent formulation of the theorem is that for $f \in \Lambda^*(N)$ of weight k , the series $\langle f \rangle_{q_{\widehat{N}}}$ is a quasimodular form of weight k and level \widehat{N} , where $q_{\widehat{N}} := q^{1/\widehat{N}}$. Secondly, as $\Gamma_1(\widehat{N}^2) \leq m_{\widehat{N}}^{-1} \Gamma(\widehat{N}) m_{\widehat{N}}$, it follows that $\langle f \rangle_q$ is a quasimodular form for $\Gamma_1(\widehat{N}^2)$.

Moreover, the definition of \widehat{N} indicates that the behaviour of $Q_k(a)$ is different when the numerator of a is divisible by 2. We will see that this can also be explained by the n -point functions F_n : they are quasi-Jacobi forms for which the index is an element of $\mathcal{M}_n(\frac{1}{2}\mathbb{Z})$.

The following theorem is a refinement of Theorem 1.2, giving us q -brackets (or quotients of q -brackets) that are quasimodular forms on $\Gamma_1(N)$ rather than only on the much smaller group $m_{\widehat{N}}^{-1} \Gamma(\widehat{N}) m_{\widehat{N}}$.

Theorem 1.3 *Let $N \geq 1$. Given $k_i \in \mathbb{Z}_{>0}, a_i \in \frac{1}{N}\mathbb{Z}$ for $i = 1, \dots, n$, denote $a = a_1 + \dots + a_n$ and $Q_k(\cdot, \mathbf{a}) = Q_{k_1}(\cdot, a_1) \cdots Q_{k_n}(\cdot, a_n)$. Then,*

- If $a \in \mathbb{Z}$, then $\langle Q_k(\cdot, a) \rangle_q$ is a quasimodular form for $\Gamma_1(N)$;
- If $a \notin \mathbb{Z}$, then $\frac{\langle Q_k(\cdot, a) \rangle_q}{\langle Q_1(\cdot, a) \rangle_q}$ is a quasimodular form for $\Gamma_1(N)$.

Remark Let $a \in \mathbb{Q}$. The function $\langle Q_1(\cdot, a) \rangle_q^{-1}$, equal to $\Theta(a)$, is a so-called *Klein form*; see e.g. [16]. Note $\langle Q_1(\cdot, a) \rangle_q = 0$ for $a \in \mathbb{Z}$.

Theorem 1.2 should be compared with the results in [12], where Griffin, Jameson and Trebat-Leder consider certain functions $Q_k^{(p)}$ in the context of studying p -adic analogues of the shifted symmetric functions (1). Extending their definition to composite m , we let

$$Q_k^{(m)}(\lambda) := Q_k(\lambda) - \frac{1}{m} \sum_{a=0}^{m-1} e\left(\frac{a}{m}\right) Q_k\left(\lambda, \frac{2a}{m}\right) \quad (k, m \geq 1). \tag{6}$$

Write $\Lambda^{(N)}$ for the graded \mathbb{Q} -algebra generated by the functions $Q_k^{(m)}$ for all $m \mid N$. For primes p , these authors show that the q -bracket of the functions $Q_k^{(p)}$ is quasimodular for $\Gamma_0(p^2)$ and they suggest that it is likely that products of these functions also have quasimodular q -brackets for the same group. Slightly more general, one can wonder whether for $f \in \Lambda^{(N)}$ the q -bracket $\langle f \rangle_q$ is a quasimodular form for $\Gamma_0(N^2)$. Now, observe that the functions $Q_k^{(m)}$ are elements of $\Lambda^*(m)$. Hence, $\Lambda^{(N)} \subset \Lambda^*(N)$. Therefore, for all odd N Theorem 1.2 implies that q -brackets of elements of $\Lambda^{(N)}$ are quasimodular for $\Gamma_1(N^2)$. That these q -brackets are indeed quasimodular for the bigger group $\Gamma_0(N^2)$ is the content of the next theorem.

Theorem 1.4 *Let $N \geq 1$ and $k \in \mathbb{Z}$. For all homogeneous $f \in \Lambda^{(N)}$ of weight k the function $\langle f \rangle_q$ is a quasimodular form of weight k for $\Gamma_0(N^2)$.*

It should be noted that the above theorems are true in greater generality. For example, the so-called hook-length moments introduced in [6], and studied in the context of harmonic Maass forms for a congruence subgroup in [3], also have natural generalisations obtained by studying their corresponding n -point functions. Similarly, the moment functions and their generalisations in [22] can equally well be generalized to congruence subgroups. Therefore, we will state and prove the above results in Sect. 3.1 in a more general setting that allows application to the hook-length moments and moment functions.

1.2 When is the q -bracket modular?

To illustrate the main ideas, consider again the Bloch–Okounkov algebra $\Lambda^* = \Lambda^*(1)$. For all $k \geq 0$, let $h_k \in \Lambda^*$ be given by

$$h_k(\lambda) := \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} \frac{Q_2(\lambda)^r Q_{k-2r}(\lambda)}{2^r (k-r-\frac{3}{2})_r r!} \quad ((x)_n := x(x+1) \cdots (x+n-1)).$$

We show that the function h_k satisfies the following three properties:

- (i) The difference $h_k - Q_k$ is divisible by Q_2 ;
- (ii) The q -bracket $\langle h_k \rangle_q$ is a modular form (and not just a quasimodular form);
- (iii) For $f \in \Lambda^*$ with $\langle f \rangle_q$ modular and $f - Q_k$ divisible by Q_2 , we have $\langle f \rangle_q = \langle h_k \rangle_q$.

Hence, one can think of the difference $h_k - Q_k$ as a correction term for Q_k with respect to the property of being a modular form under the q -bracket. By the third property, this

correction term is unique up to elements in the kernel of the q -bracket. More generally, one has the following result. Here, M denotes the algebra of modular forms for $SL_2(\mathbb{Z})$ with rational Fourier coefficients.

Theorem 1.5 *For any algebra \mathcal{F} of functions on partitions satisfying the conditions in Sect. 4.1, there exists a computable subspace $\mathcal{M} = \mathcal{M}(\mathcal{F}) \subseteq \mathcal{F}$ such that*

- (i) $\mathcal{F} = \mathcal{M} \oplus Q_2\mathcal{F}$;
- (ii) $\langle \mathcal{M} \rangle_q \subseteq M$;
- (iii) $\langle Q_2\mathcal{F} \rangle_q \cap M = \{0\}$.

The algebra $\mathcal{F} = \Lambda^*$ is an example to which the above result applies. In particular, by using (1.5) inductively, each $f \in \Lambda^*$ can uniquely be written as a polynomial in Q_2 with coefficients $g_i \in \mathcal{M}$, i.e.

$$f = \sum_{i \geq 0} g_i Q_2^i.$$

As $\langle g \rangle_q = 0$ if and only if $\langle Q_2 g \rangle_q = 0$ for $g \in \Lambda^*$, the following are equivalent:

- (a) $\langle f \rangle_q$ is modular;
- (b) $\langle f \rangle_q = \langle g_0 \rangle_q$;
- (c) $\langle g_i \rangle_q = 0$ for all $i > 0$.

The functions $h_\lambda \in \Lambda^*$ defined in [21] form a basis for the space $\mathcal{M}(\Lambda^*)$. The method of proof in this work (i.e. using quasi-Jacobi forms) allowed us to state Theorem 1.5 for many algebras \mathcal{F} , whereas the results in [21] could not easily be generalized to other algebras than Λ^* . In Sect. 4, we prove this theorem and apply this result to the algebra Λ^* of shifted symmetric functions, to its extensions to higher levels $\Lambda^*(N)$ and to the aforementioned algebra of moment functions.

1.3 Quasi-Jacobi forms and their Taylor coefficients

Recall $\tau \in \mathfrak{h}$, the complex upper half plane, $z \in \mathbb{C}$ and $e(x) = e^{2\pi ix}$, $q = e(\tau)$. The Kronecker–Eisenstein series E_k are given by

$$E_k(\tau, z) := \sum_{m,n \in \mathbb{Z}}^e \frac{1}{(z + m\tau + n)^k} \quad (k \geq 1) \tag{7}$$

Here, the letter ‘e’ indicates we perform the Eisenstein summation procedure, given by $\sum_{m,n \in \mathbb{Z}}^e := \lim_{M \rightarrow \infty} \sum_{m=-M}^M (\lim_{N \rightarrow \infty} \sum_{n=-N}^N)$. Note this summation procedure is only important for $k = 1$ and $k = 2$, as for $k \geq 3$ the series converges absolutely.

Do these series E_k , the Jacobi theta function θ and its derivatives, and the Bloch–Okounkov n -points functions F_n have a common property?

To answer this question, observe that the first two functions can be interpreted as generating functions of the Eisenstein series e_k , given by¹

$$e_k(\tau) := \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k} \quad (k \geq 2).$$

¹Here, we follow the notation in [23], which goes back to Eisenstein. Often, e_k is denoted by G_k .

Namely,

$$\Theta(\tau, z) := \frac{\theta(\tau, z)}{\theta'(\tau, 0)} = 2\pi iz \exp\left(-\sum_{m \geq 1} \frac{e_{2m}(\tau)}{2m} z^{2m}\right),$$

$$E_k(\tau, z) = \frac{1}{z^k} + (-1)^k \sum_{m \geq k/2} \binom{2m-1}{k-1} e_{2m}(\tau) z^{2m-k}.$$

In particular, the Taylor (or Laurent) coefficients around $z = 0$ of these functions are polynomials in Eisenstein series, hence quasimodular forms—a property which is shared by the stronger notion of a Jacobi form. Just as e_2 transforms as a *quasimodular* form, so do $\theta^{(r)}$ and F_n transform as *quasi-Jacobi* forms. Our answer to the question is that all these functions are quasi-Jacobi forms, introduced in Sect. 2.4.

Quasi-Jacobi forms transform comparable to quasimodular forms, as we explain now. There is a slash action (see Definition 2.2) on all functions $\varphi: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$ and for all $X \in M_{n,2}(\mathbb{Q})$ (so actually for the action of their semidirect product $\text{SL}_2(\mathbb{Z}) \times M_{n,2}(\mathbb{Q})$). In case φ is a quasi-Jacobi form of weight k and index M , there exist quasi-Jacobi forms $\varphi_{i,j}$, indexed by a finite subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n$, such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = (\lambda, \mu) \in M_{n,2}(\mathbb{Z})$ one has

$$(\varphi|_{k,M} \gamma)(\tau, z_1, \dots, z_n) = \sum_{i,j} \varphi_{i,j}(\tau, z_1, \dots, z_n) \left(\frac{c}{c\tau + d}\right)^{i+|j|} \frac{z_1^{j_1} \cdots z_n^{j_n}}{(2\pi i)^i}, \tag{8}$$

$$(\varphi|_M X)(\tau, z_1, \dots, z_n) = \sum_j \varphi_{0,j}(\tau, z_1, \dots, z_n) (-\lambda_1)^{j_1} \cdots (-\lambda_n)^{j_n} \tag{9}$$

together with similar formulas for each $\varphi_{i,j}|_{k,M} \gamma$ and $\varphi_{i,j}|_M X$. Jacobi forms are quasi-Jacobi forms for which the right-hand side in both equations above equals φ .

The quasi-Jacobi forms φ we study are *strictly meromorphic*, i.e. meromorphic such that if $\mathbf{z} \in \mathbb{R}^n \tau + \mathbb{R}^n$ is a pole of $\varphi(\tau, \cdot)$ for some $\tau \in \mathfrak{h}$, it is a pole for almost all $\tau \in \mathfrak{h}$. This is a new notion we introduce in this work; see Sect. 2.1. The Weierstrass \wp -function is an example of a strictly meromorphic Jacobi form, but its inverse is not. Also, E_k, Θ^{-1} and F_n are strictly meromorphic. If the number of elliptic variables n satisfies $n = 1$, this condition is equivalent to the statement that all poles of φ are torsion points $z \in \mathbb{Q}\tau + \mathbb{Q}$. This is crucial to obtain mock modular forms as Fourier coefficients of meromorphic Jacobi forms (see [8,26]). For $n > 1$, the Jacobi transformation properties of φ imply a more complicated restriction on the positions of the poles. By studying the orbits of the action of $\text{SL}_2(\mathbb{Z})$ on $(\mathbb{R}^2)^n$ we prove the following result.

Theorem 1.6 *Let φ be a strictly meromorphic quasi-Jacobi form and $\tau \in \mathfrak{h}$. Then, all poles \mathbf{z} of $\varphi(\tau, \cdot)$ lie in finite union of rational hyperplanes*

$$s_1 z_1 + \dots + s_n z_n \in u\tau + v$$

with $s_1, \dots, s_n \in \mathbb{Z}$ and $u, v \in \mathbb{Q}/\mathbb{Z}$.

In this work, we determine conditions on the Taylor coefficients (or rather Laurent coefficients in case we are expanding around a pole) of a meromorphic function $\varphi: \mathfrak{h} \times$

$\mathbb{C}^n \rightarrow \mathbb{C}$ for it to be a (quasi-)Jacobi form. This is the technical result we need to answer Question I and II. In one direction, by the work of Eichler and Zagier [9], it is known that for a Jacobi form the Taylor coefficients around rational lattice points are quasimodular, or equivalently that certain linear combinations ξ_ℓ^X of derivatives of these Taylor coefficients are modular. For example, Θ is a weak Jacobi form, hence it satisfies

$$(\Theta|X|\gamma)(\tau, z) \propto (\Theta|X\gamma)(\tau, z)$$

for all $X = (\lambda, \mu) \in M_{1,2}(\mathbb{Q})$ and $\gamma \in \text{SL}_2(\mathbb{Z})$, where the implicit multiplicative constant is a root of unity depending on X and γ . Hence, it follows that the Taylor coefficients of Θ around $z = \lambda\tau + \mu$ (after multiplying with a certain power of q) are quasimodular for some subgroup Γ_X of $\text{SL}_2(\mathbb{Z})$ consisting of γ for which $X\gamma - X \in M_{1,2}(\mathbb{Z})$ (see 18). In contrast, the weak quasi-Jacobi form $\Theta'(\tau, z) = \frac{\theta'(\tau, z)}{\theta'(\tau, 0)}$ transforms as

$$(\Theta'|X|\gamma)(\tau, z) \propto (\Theta'|X\gamma)(\tau, z) + \frac{cz}{c\tau + d}(\Theta|X\gamma)(\tau, z) + \lambda(\Theta|X\gamma)(\tau, z) - \frac{\lambda}{c\tau + d}(\Theta|X\gamma)(\tau, z).$$

Therefore, the Taylor coefficients of $\Theta' + \lambda\Theta$ around $z = \lambda\tau + \mu$, rather than of Θ' , give rise to quasimodular forms for Γ_X . We write $g_\ell^X(\Theta')$ for the ℓ th Taylor coefficient of $\Theta' + \lambda\Theta$. In Definition 2.29 we extend this notation by writing $g_\ell^X(\varphi)$ for the ‘correct’ Taylor coefficient of a quasi-Jacobi form φ . Moreover, in (20) we define $\xi_m^X(\varphi)$ to be a certain combination of the derivatives of $g_\ell^X(\varphi)$.

The main result on Taylor coefficients of quasi-Jacobi forms is given by Theorem 2.35 and summarized in the following result. For simplicity, we assume that $\mathbf{s} = (s_1, \dots, s_n)$ in Theorem 1.6 is always a standard basis vector, e.g. we allow $\wp(z_1)\wp(z_2 + \frac{1}{2})$, but we do not allow $\wp(z_1 - z_2)$.

Theorem 1.7 *Let φ be a strictly meromorphic quasi-Jacobi form of weight k and index M whose poles \mathbf{z} lie on a finite collection of hyperplanes of the form $z_j \in u\tau + v$ with $j \in \{1, \dots, n\}$ and $u, v \in \mathbb{Q}/\mathbb{Z}$. Then*

- (i) *for all $X \in M_{n,2}(\mathbb{Q})$ and $\ell \in \mathbb{Z}^n$ the ‘Taylor coefficients’ $g_\ell^X(\varphi)$ are quasimodular forms of weight $k + \ell_1 + \dots + \ell_n$ for the group Γ_X and satisfy the functional Eqs. (21) and (22).*
- (ii) *for all $X \in M_{n,2}(\mathbb{Q})$ and $\mathbf{m} \in \mathbb{Z}^n$ the functions $\xi_m^X(\varphi)$ are modular forms of weight $k + m_1 + \dots + m_n$ for Γ_X and satisfy the functional Eq. (21).*

Definition 2.29 and Eq. (20) (defining $g_\ell^X(\varphi)$ and $\xi_m^X(\varphi)$) use the functions $\varphi_{i,j}$ in (8) and (9), which are uniquely determined by the quasi-Jacobi form φ . The results of Sect. 2.7 also show that the four collections of functions $\{\varphi\}$, $\{\varphi_{i,j}\}$, $\{g_\ell^X\}$ and $\{\xi_m^X\}$ determine each other in a computable way, and give explicit conditions on the collections $\{\varphi_{i,j}\}$, $\{g_\ell^X\}$ and $\{\xi_m^X\}$ that imply that they arise from a quasi-Jacobi form.

In Sect. 2, we introduce strictly meromorphic quasi-Jacobi forms and prove Theorem 1.6 and Theorem 1.7. The quasimodular transformation of the Taylor coefficients of F_n will then imply the results on the Bloch–Okounkov theorem for congruence subgroups in Sect. 3. Moreover, in Sect. 4, we see that “pulling back” the functions ξ_m^X under the q -bracket leads to the construction of the functions h_m for which the q -bracket is modular.

2 Quasi-Jacobi forms

The definition of a Jacobi form in [9] has been generalized in many ways. We provide a generalization that incorporates several elliptic variables, characters, weakly holomorphic and meromorphic functions, and quasi-Jacobi forms [2, 19, 20]. In particular, the normalized Jacobi theta function Θ , the Kronecker–Eisenstein series E_k and the Bloch–Okounkov n -point functions F_n from the introduction will be examples throughout this paper.

For $\tau \in \mathfrak{h}$, the complex upper half plane, we write $L_\tau = \mathbb{Z}\tau + \mathbb{Z}$. As is customary, we often omit the dependence on the modular variable τ in any type of Jacobi form, e.g. we write $\Theta(z)$ for $\Theta(\tau, z)$. We write $\mathbf{a} = (a_1, \dots, a_n)$ for a vector of elements and we write $|\mathbf{a}|$ for $a_1 + \dots + a_n$. For vectors \mathbf{a} and \mathbf{b} , we write $\mathbf{a}^{\mathbf{b}}$ to denote $\prod_i a_i^{b_i}$. Also, given $X \in M_{n,2}(\mathbb{R})$, we denote the rows of X by λ and μ , i.e. $X = (\lambda, \mu)$. Moreover, for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and we write λ^γ and μ^γ for the rows of $X\gamma$, i.e. $X\gamma = (\lambda^\gamma, \mu^\gamma)$.

2.1 Strictly meromorphic Jacobi forms

The definition of a strictly meromorphic Jacobi form is subtle, excluding many meromorphic functions transforming as Jacobi forms. For example, the j -invariant, the reciprocal of an Eisenstein series e_k , or \wp/Δ , where \wp is the Weierstrass \wp -function and Δ the modular discriminant, are all *non-examples* of strictly meromorphic Jacobi forms. Namely, although a strictly meromorphic Jacobi form is meromorphic we want its Taylor coefficients in the elliptic variables to be holomorphic (rather than weakly holomorphic or meromorphic) quasimodular forms. But with this not everything has been said, the definition is even stricter: we require the poles of φ to be “constant” in the modular variable τ . Consider, for example, the Weierstrass \wp -function, which is an example of a strictly meromorphic Jacobi form. For every fixed $z \in \mathbb{C}$ (e.g. $z = i$) the function $\wp(\tau, z)$ is a meromorphic function of τ , as $\wp(\tau, z)$ has a pole whenever τ is such that $z \in L_\tau$ (e.g. $\tau = z = i$). However, for $\lambda, \mu \in \mathbb{R}$, the function $\wp(\tau, \lambda\tau + \mu)$ is holomorphic, unless both $\lambda \in \mathbb{Z}$ and $\mu \in \mathbb{Z}$. That is, all poles of $\wp(\tau, z)$ are given by $z \in L_\tau$. On the contrary, we will see that \wp^{-1} is not a strictly meromorphic Jacobi form, as the poles of \wp^{-1} are not “constant” in τ .

Before introducing strictly meromorphic Jacobi forms, we first recall the Jacobi group and its action on (meromorphic) functions.

Definition 2.1 For all $n \in \mathbb{N}$, the (discrete) Jacobi group Γ_n^J of rank n is defined as the semi-direct product $\Gamma_n^J := \mathrm{SL}_2(\mathbb{Z}) \ltimes M_{n,2}(\mathbb{Z})$ with respect to the right action of $\mathrm{SL}_2(\mathbb{Z})$ on $M_{n,2}(\mathbb{Z})$.

That is, an element of Γ_n^J is a pair (γ, X) with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $X \in M_{n,2}(\mathbb{Z})$ and satisfies the group law $(\gamma, X)(\gamma', X') = (\gamma\gamma', X + X\gamma')$.

Let $M \in M_n(\mathbb{Q})$. We often make use of the associated bilinear form B_M , given by²

$$B_M(\mathbf{z}, \mathbf{z}') = \mathbf{z}M\mathbf{z}'^t.$$

Definition 2.2 Given a meromorphic function $\varphi: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$, $k \in \mathbb{Z}$ and $M \in M_n(\mathbb{Q})$, for all $(\gamma, X) \in \Gamma_n^J$ we let

²It is standard to let $Q_M(\mathbf{z}) = \frac{1}{2}\mathbf{z}M\mathbf{z}'^t$ be the associated quadratic form. We refrain from using this notation to avoid a clash with the shifted symmetric functions denoted by Q_k .

- (i) $(\varphi|_{k,M}\gamma)(\tau, \mathbf{z}) := (c\tau + d)^{-k} e\left(\frac{-c B_M(\mathbf{z}, \mathbf{z})}{c\tau + d}\right) \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathbf{z}}{c\tau + d}\right);$
- (ii) $(\varphi|_M X)(\tau, \mathbf{z}) := e(B_M(\boldsymbol{\lambda} + \boldsymbol{\mu}, \boldsymbol{\lambda} + \boldsymbol{\mu})) e(B_M(\boldsymbol{\lambda}, \boldsymbol{\lambda}\tau + 2\mathbf{z})) \varphi(\tau, \mathbf{z} + \boldsymbol{\lambda}\tau + \boldsymbol{\mu}).$

(Recall that we write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = (\boldsymbol{\lambda}, \boldsymbol{\mu}).$) Moreover, we let $\varphi|_{k,M}(\gamma, X) := (\varphi|_{k,M}\gamma)|_M X$, often omitting k and M from the notation.

Remark Given $k \in \mathbb{Z}$ and $M \in M_n(\mathbb{Q})$, the slash operator defines an action of Γ_n^J of weight k and index M on the space of all meromorphic functions $\varphi: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}.$

Given $M \in M_n(\mathbb{Q})$, for $X, X' \in M_{n,2}(\mathbb{Q})$, we let

$$\rho(X) := e(B(\boldsymbol{\lambda}, \boldsymbol{\lambda}) - B(\boldsymbol{\lambda}, \boldsymbol{\mu}) + B(\boldsymbol{\mu}, \boldsymbol{\mu})), \quad \zeta_{X,X'} := e(B(\boldsymbol{\lambda}', \boldsymbol{\mu}) - B(\boldsymbol{\lambda}, \boldsymbol{\mu}')), \tag{10}$$

where we wrote $B = B_M$ for the bilinear form associated to $M.$ Observe that

$$\rho(-X) = \rho(X)^{-1} \quad \text{and} \quad \zeta_{X',X} = \zeta_{X,X'}^{-1} = \zeta_{-X,X'} = \zeta_{X,-X'}.$$

By extending the slash action to the real Jacobi group, generalizing [9, Theorem 1.4] to several variables and half-integral index, we obtain the following functional equations.

Proposition 2.3 *Given a meromorphic function $\varphi: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}, k \in \mathbb{Z}$ and $M \in M_n(\mathbb{Q})$, for all $X, X' \in M_{n,2}(\mathbb{R})$ and $\gamma \in \text{SL}_2(\mathbb{Z})$ one has*

$$\rho(-X) \varphi|_X | \gamma = \rho(-X\gamma) \varphi|_{\gamma|X\gamma}$$

and

$$\begin{aligned} \rho(-X) \rho(-X') \zeta_{X',X} \varphi|_X | X' &= \rho(-X') \rho(-X) \zeta_{X,X'} \varphi|_{X'|X} \\ &= \rho(-X - X') \varphi|(X + X'). \end{aligned}$$

Classical modular forms are defined as the invariants for a certain group action of the space $\text{Hol}_0(\mathfrak{h})$ of holomorphic functions in \mathfrak{h} satisfying a certain growth condition (having at most polynomial growth near the boundary).

Definition 2.4 Let $\text{Hol}_0(\mathfrak{h})$ be the ring of holomorphic functions φ of moderate growth on \mathfrak{h} , i.e. for all $C > 0, \gamma \in \text{SL}_2(\mathbb{Z})$ and $x \in \mathbb{R}$ one has $\varphi(\gamma(x + iy)) = O(e^{Cy})$ as $y \rightarrow \infty$ (where γ acts on \mathfrak{h} by Möbius transformations).

With the remarks at the beginning of this section in mind, we now define strictly meromorphic Jacobi forms. A final subtlety in the definition below is coming from the fact that a meromorphic function in two or more variables always has points of indeterminacy (think of x/y near the origin, whose limiting value depends on the angle of approach). Points of indeterminacy are not “generic”, and we exclude these points when we say, for instance, that a certain function $\varphi(\tau, \mathbf{z})$ has its poles precisely on certain hyperplanes for all *generic* $\tau \in \mathfrak{h}.$

Definition 2.5 Given $n \geq 0$, denote by Mer_n the space of meromorphic functions $\varphi: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$ such that for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^n$ either $\mathbf{z} = \boldsymbol{\lambda}\tau + \boldsymbol{\mu}$ is a pole of $\varphi(\tau, \cdot)$ for all generic $\tau \in \mathfrak{h}$ or the function $\tau \mapsto \varphi(\tau, \boldsymbol{\lambda}\tau + \boldsymbol{\mu})$ belongs to $\text{Hol}_0(\mathfrak{h}).$ Moreover, given

$M \in M_n(\mathbb{Q})$, denote by Mer_n^M the subspace of $\varphi \in \text{Mer}_n$ for which $\varphi|_M X \in \text{Mer}_n$ for all $X \in M_{n,2}(\mathbb{Q})$. Let Hol_n and Hol_n^M be the subspace in Mer_n and Mer_n^M , respectively, of holomorphic functions.

Definition 2.6 Let $k \in \mathbb{Z}$ and $M \in M_n(\mathbb{Q})$. A *holomorphic, weak, or a strictly meromorphic Jacobi form* of weight k , index M and rank n for the Jacobi group Γ_n^J is a function φ in Hol_n^M , Hol_n or Mer_n^M , respectively, that is invariant under the action of Γ_n^J of weight k and index M (i.e. $\varphi|_{k,M}g = \varphi$ for all $g \in \Gamma_n^J$). We write $J_{k,M}^{\text{hol}}$, $J_{k,M}^{\text{weak}}$ and $J_{k,M}^{\text{sm}}$ for the vector spaces of holomorphic, weak, and strictly meromorphic Jacobi forms of weight k and index M (often omitting the indices).

Remark Let $M = (m_{ij}) \in M_n(\mathbb{Q})$. Any space of Jacobi forms is trivial whenever $2B_M(\mathbf{z}, \mathbf{z})$ is a non-integral quadratic form, or, equivalently, when $m_{ij} \notin \frac{1}{4}\mathbb{Z}$ or $m_{ii} \notin \frac{1}{2}\mathbb{Z}$ for some $i \neq j$. Namely, let φ be a Jacobi form and let $\tau \in \mathfrak{h}$ be fixed. If φ is nonzero of rank 1, write $M = (m)$ for its index. It follows from the elliptic transformation law (Definition 2.2(ii)) that the number of zeros minus the number of poles of $z \mapsto \varphi(\tau, z)$ in any fundamental domain for the action of L_τ on \mathbb{C} is exactly $2m$. For Jacobi forms of higher rank the integrability of $2B_M(\mathbf{z}, \mathbf{z})$ follows by noting that for fixed $\mu_2, \dots, \mu_n \in \mathbb{C}$, functions of the form $z \mapsto \varphi(\tau, z, \mu_2, \mu_3, \dots, \mu_n)$ and $z \mapsto \varphi(\tau, z, z, \mu_3, \dots, \mu_n)$ still satisfy the elliptic transformation law.

A holomorphic Jacobi form of rank 0 is just a modular form. More interestingly, the Kronecker–Eisenstein series (7) are examples of a strictly meromorphic Jacobi form of index (0), with expansions given by

$$E_k(\tau, z) = \frac{(-1)^k}{(k-1)!} D_y^{k-2} \left(\sum_{m \in \mathbb{Z}} \frac{yq^m}{(1-yq^m)^2} \right) \quad \left(y = e(z), D_y = y \frac{\partial}{\partial y} \right).$$

Closely related are the Weierstrass \wp -function and its derivative, that is, $\wp = E_2 - e_2$ and $\wp' = -2E_3$. By [9, Theorem 9.4], it follows that the algebra of weak Jacobi forms is given by

$$J^{\text{weak}} = \mathbb{C}[A, B, C, e_4, e_6] / (C^2 - 4AB^3 + 60e_4A^3B + 140e_6A^4),$$

where A, B and C are equal to Θ^2 , $\wp \Theta^2$ and $\wp' \Theta^4$, respectively. Note that the relation $C^2 = 4AB^3 - 60e_4A^3B - 140e_6A^4$ comes from the differential equation satisfied by the Weierstrass \wp -function.

The following result, yielding an algebraic proof and extending [19, Proposition 2.8]³, gives all strictly meromorphic Jacobi forms of rank 1 with only poles at the lattice points. The corresponding algebra is free, as the relation between \wp^3 and $(\wp')^2$ can be used to express e_6 in terms of the generators. As usual, we write m (instead of the matrix $M = (m)$) for the index of a Jacobi form of rank 1.

Proposition 2.7 Let $\varphi \in J^{\text{sm}}$ be of index $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ such that all poles (τ, z) of φ satisfy $z \in L_\tau$. Then,

$$\varphi \in \mathbb{C}[\wp, \wp', e_4] \Theta^{2m}.$$

³The author states the result for *Jacobi forms*, which obviously is not meant to be holomorphic Jacobi forms. Although not stated explicitly, we assume he refers to strictly meromorphic Jacobi forms with poles only at lattice points.

Proof First, we show that $f = \varphi \Theta^{-2m} \in J^{\text{sm}}$ is a strictly meromorphic Jacobi form of index 0 with all poles (τ, z) satisfying $z \in L_\tau$. This follows from the claim that $\Theta^{-1} \in J_{1,-1/2}^{\text{sm}}$ and that all the poles of Θ^{-1} are at the lattice points. To prove this claim, note that by the Jacobi triple product

$$\Theta = (y^{1/2} - y^{-1/2}) \prod_{n \geq 1} \frac{(1 - yq^n)(1 - y^{-1}q^n)}{(1 - q^n)^2}.$$

It follows that Θ is a weak Jacobi form with all zeros at the lattice points $z \in L_\tau$. Moreover, for all $X \in M_{n,2}(\mathbb{Q})$ the function $\Theta|X$ does not vanish at infinity, from which the claim follows.

From now on, assume that φ is of index 0, i.e. that φ is an elliptic function. Write $\varphi = \varphi_0 + \varphi_1$ with φ_0 and φ_1 the even and odd part of φ , respectively. For $u, v \in \mathbb{R}\tau + \mathbb{R}$, write $u \sim v$ if $u \equiv v$ or $u \equiv -v \pmod{L_\tau}$. Then, for $i = 0, 1$ one has

$$\varphi_i = (\wp')^i \prod_j (\wp - \wp(u_j(\tau)))^{m_j},$$

where $m_j \in \mathbb{Z}$ and $u_j(\tau)$ are representatives with respect to the above equivalence relation for the zeros and poles of φ_i outside L_τ . As both φ_0 and φ_1 do not admit poles outside the lattice, it follows that $m_j > 0$. Hence, φ is a polynomial in \wp and \wp' where the coefficients are polynomials in the functions $\wp(u_j(\tau))$. By the modular transformation every such coefficient is a modular form for $\text{SL}_2(\mathbb{Z})$, hence an element of $\mathbb{C}[\wp, \wp', e_4]$.

Remark Although the above result and many examples of strictly meromorphic Jacobi forms in the literature have only poles at $z \in L_\tau$, one easily constructs a strictly meromorphic Jacobi forms with poles at different places. Namely, if φ is a Jacobi form with all poles at $z \in L_\tau$, then

$$\varphi(\tau, z + \frac{1}{2}) + \varphi(\tau, z + \frac{1}{2}\tau) + \varphi(\tau, z + \frac{1}{2}\tau + \frac{1}{2})$$

is a Jacobi form for the same group, but now with the poles at $\frac{1}{2}, \frac{1}{2} + \frac{1}{2}\tau$ and $\frac{1}{2}\tau$ modulo the lattice L_τ .

2.2 Poles of Jacobi forms

In contrast to the space of (weakly) holomorphic Jacobi forms, the space of strictly meromorphic Jacobi forms of given weight and index is not finite-dimensional. However, the latter space is not far from being finite-dimensional. First of all, in contrast to the space of all meromorphic functions, the space of strictly meromorphic Jacobi forms is not a field. Moreover, the poles lie in a finite number of hyperplanes and after fixing finitely many such hyperplanes to contain the poles, the vector space of strictly meromorphic Jacobi forms of given weight and index is finite-dimensional, as we will explain in this section.

Given a meromorphic Jacobi form φ of rank n , we write

$$P_\varphi = \{(\tau, \mathbf{z}) \in \mathfrak{h} \times \mathbb{C}^n \mid \varphi \text{ is not holomorphic at } (\tau, \mathbf{z})\}$$

for the set of poles as well as points of indeterminacy of φ . We identify two points of P_φ if they have same image under the projection $\mathfrak{h} \times \mathbb{C}^n \rightarrow M_{n,2}(\mathbb{R})$ given by $(\tau, \lambda\tau + \boldsymbol{\mu}) \mapsto (\lambda, \boldsymbol{\mu})$. That is, we define an equivalence relation on P_φ by saying that $(\tau, \mathbf{z}) \sim (\tau', \mathbf{z}')$ whenever, after writing $\mathbf{z} = \lambda\tau + \boldsymbol{\mu}$ and $\mathbf{z}' = \lambda'\tau + \boldsymbol{\mu}'$ with $\lambda, \lambda', \boldsymbol{\mu}, \boldsymbol{\mu}' \in \mathbb{R}^n$, one has $\lambda = \lambda'$ and $\boldsymbol{\mu} = \boldsymbol{\mu}'$. We identify the quotient set Q_φ with a subset of $M_{n,2}(\mathbb{R})$ by

identifying a point of P_φ with its image under the projection. From the definition of a strictly meromorphic Jacobi form, we obtain the factorisation $\mathfrak{h} \times Q_\varphi \simeq P_\varphi$ of P_φ , given by $(\tau, (\lambda, \mu)) \mapsto (\tau, \lambda\tau + \mu)$. Note that Q_φ is invariant under translation by $M_{n,2}(\mathbb{Z})$.

As an example of how the definition works, we first prove a simple consequence:

Proposition 2.8 *Let $\varphi \in J^{\text{sm}}$. If $\frac{1}{\varphi}$ is also a strictly meromorphic Jacobi form, then φ is constant.*

Proof Let $X = (\lambda, \mu) \in M_{1,2}(\mathbb{R})$ be given with $X \notin Q_\varphi$ and $X \notin Q_{1/\varphi}$. Write $\mathbf{z}(\tau) = X(\tau, 1)^\tau = \lambda\tau + \mu$. Then both $\varphi(\tau, \mathbf{z}(\tau))$ and $\frac{1}{\varphi(\tau, \mathbf{z}(\tau))}$ are holomorphic as a function of $\tau \in \mathfrak{h}$. Hence, as a function of $\tau \in \mathfrak{h}$, both $\varphi(\tau, \mathbf{z}(\tau))$ and $\frac{1}{\varphi(\tau, \mathbf{z}(\tau))}$ do not admit any zeros. Similarly, it follows that both $\varphi(\tau, \mathbf{z}(\tau))$ and $\frac{1}{\varphi(\tau, \mathbf{z}(\tau))}$ are holomorphic (as a function of $\tau \in \mathfrak{h}$) at the cusps. Hence, both functions do not admit any zero at the cusps. In other words, $\varphi(\tau, \mathbf{z}(\tau))$ does not admit any zeros and poles on a compact set. So, $\varphi(\tau, \mathbf{z}(\tau))$ is constant as a function of τ . As this holds for almost all $X \in M_{n,2}(\mathbb{R})$, we conclude that φ is globally constant.

The following result, which is crucial for the sequel, tells us that the image of Q_φ in the torus $M_{n,2}(\mathbb{R}/\mathbb{Z})$ consists of finitely many hyperplanes given by linear equations with rational coefficients. In other words, the following result strengthens Theorem 1.6 (the fact that the second conclusion is also true for quasi-Jacobi forms, follows immediately after introducing such functions (see Corollary 2.19).

Theorem 2.9 *Let $\varphi \in J^{\text{sm}}$, and let $P_\varphi \simeq \mathfrak{h} \times Q_\varphi$ be the set of non-holomorphic points as above. Then, we have:*

- (i) *If $X \in Q_\varphi$, then $X\gamma \in Q_\varphi$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$.*
- (ii) *There exist finitely many hyperplanes of the form*

$$s \cdot X \in (u, v)$$

with $s \in \mathbb{Z}^n$ primitive (i.e. with coprime entries) and $u, v \in \mathbb{Q}/\mathbb{Z}$, such that $X \in Q_\varphi$ precisely if X lies on such a hyperplane.

Proof (2.9): Let $X \in Q_\varphi$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ be given. Write $\mathbf{x}(\tau) = X(\tau, 1)^\tau = \lambda\tau + \mu$. Note that φ has a pole at $\mathbf{x}(\tau)$ for generic $\tau \in \mathfrak{h}$. By the modular transformation behaviour of φ under γ^{-1} , it follows that

$$\varphi\left(\frac{d\tau - b}{-c\tau + a}, \frac{\mathbf{z}}{-c\tau + a}\right)$$

also has a pole at $\mathbf{z} = \mathbf{x}(\tau)$ for generic $\tau \in \mathfrak{h}$. Now, let $\tau' = \gamma^{-1}\tau$. We find

$$\frac{\mathbf{x}(\tau)}{-c\tau + a} = \frac{\mathbf{x}(\gamma\tau')}{-c\gamma\tau' + a} = \frac{\lambda\gamma\tau' + \mu}{-c\gamma\tau' + a} = \lambda(a\tau' + b) + \mu(c\tau' + d) = (\lambda, \mu)\gamma(\tau', 1)^\tau.$$

Hence, the function $\varphi(\tau, \mathbf{z})$ has a pole at $\mathbf{z} = X\gamma(\tau, 1)^\tau$ for generic $\tau \in \mathfrak{h}$.

(2.9): Let $X = (\lambda, \mu) \in Q_\varphi$. The set Q_φ is closed and $(n - 1)$ -dimensional; hence, there is a $Y \in M_{n,2}(\mathbb{R})$ with $Y \notin Q_\varphi$ and which is bounded away from $X\gamma$.

First, we treat the case that the rank n is 1. Then, by Lemma 2.12, proven in the next section, the fact that Y is bounded away from $X\gamma$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$ implies that both λ and μ are rational. Therefore, we can take $s = 1$ and $(u, v) = (\lambda, \mu) \bmod \mathbb{Z}^2$.

Next, let $n \geq 2$. By an approximation property proven in the next section (see Proposition 2.15), the fact that Y is bounded away from X implies that there are non-trivial $\mathbf{s} \in \mathbb{Z}^n, t \in \mathbb{Z}$ with $\mathbf{s} \cdot \boldsymbol{\lambda} = t$. (In fact, the result states that if Y is bounded away from X then for all $\alpha, \beta \in \mathbb{Z}$, there exist non-trivial $\mathbf{s} \in \mathbb{Z}^n, t \in \mathbb{Z}$ with $\mathbf{s} \cdot (\alpha\boldsymbol{\lambda} + \beta\boldsymbol{\mu}) = t$. We choose $\alpha = 1, \beta = 0$.) Note that the value of \mathbf{s} and t is a function of X , i.e. for all elements $X \in Q_\varphi$ there exists an element $(\mathbf{s}, t) \in \text{Rel}$ giving the relation, where

$$\text{Rel} := (\mathbb{Z}^n \setminus \{\mathbf{0}\}) \times \mathbb{Z}.$$

Let $\tau \in \mathfrak{h}$ be given and identify $M_{n,2}(\mathbb{R})$ with \mathbb{C}^n via $X \mapsto \mathbf{x} = X(\tau, 1)^t$. For almost all $X \in Q_\varphi$, the conditions of the implicit function theorem for the function φ are satisfied. In this case, there exists an open set $U \subset \mathbb{C}^{n-1}$ containing $\mathbf{0}$, an open neighbourhood $V \subset \mathbb{C}^n$ of \mathbf{x} , and a holomorphic function $g : U \rightarrow V$, such that $\mathbf{z} \in V$ is a pole of φ precisely if \mathbf{z} lies in the image of g . Consider the isomorphism $(\pi_\tau, \rho_\tau) : \mathbb{C} \simeq \mathbb{R}\tau + \mathbb{R}$. Let g_i be the i th component function of g , which takes values in $\mathbb{R}\tau + \mathbb{R}$ under this isomorphism. For example, taking $\tau = i$, we find $\pi_i(g_i) = \text{Im}(g_i)$ and $\rho_i(g_i) = \text{Re}(g_i)$. For every $\mathbf{u} \in U$ we find a relation of the form

$$\sum_{i=1}^n s_i \pi_\tau(g_i(\mathbf{u})) = t, \tag{11}$$

where $(\mathbf{s}, t) \in \text{Rel}$, possibly depending on the choice of \mathbf{u} . We now show that we can choose $(\mathbf{s}, t) \in \text{Rel}$ such that (11) holds for all $\mathbf{u} \in U$. Recall that the set of zeros of a non-constant real-analytic function has measure 0. As $\pi_\tau(g_i)$ is a real-analytic function, either (11) holds for all $\mathbf{u} \in U$, or it holds for a real subspace of U of measure 0. Now, note that Rel is a countable set, whereas countably many subspaces of measure 0 do not cover U . Hence, we find a relation (11) which holds for all $\mathbf{u} \in U$. By the Cauchy–Riemann equations for the holomorphic functions g_i we can ‘upgrade’ this relation to the statement that $\mathbf{s} \cdot \mathbf{z}$ takes a constant value $u\tau + v$ in $\mathbb{R}\tau + \mathbb{R}$ for all $\mathbf{z} \in g(U)$. Without loss of generality we assume that $\text{gcd}(s_1, \dots, s_n) = 1$; if not, we scale \mathbf{s}, u and v appropriately.

Now, by the Weierstrass preparation theorem, locally around \mathbf{x} the set of poles is given by k branches coming together, where each branch is an $(n - 1)$ -dimensional space given by the zeros of a holomorphic function and k equals the multiplicity of the pole at \mathbf{x} (see [7, Lemma 6.1] for the same argument in a different setting). We know that almost all (hence all) the elements in such a branch satisfy $\mathbf{s} \cdot \mathbf{x} = u\tau + v$. Write T for $(\mathbb{R}\tau + \mathbb{R})/(\mathbb{Z}\tau + \mathbb{Z})$. By analytically extending such a local branch, we find that all solutions $\mathbf{z} \in T$ of $\mathbf{s} \cdot \mathbf{z} = u\tau + v$ are poles of $\varphi(\tau, \cdot)$ as long as they are in the same connected component as \mathbf{x} . Because $\text{gcd}(s_1, \dots, s_n) = 1$ the solution space of $\mathbf{s} \cdot \mathbf{z} = u\tau + v$ in T is connected, so that all solutions \mathbf{z} of $\mathbf{s} \cdot \mathbf{z} \equiv u\tau + v \pmod{L_\tau}$ are poles of $\varphi(\tau, \cdot)$ for generic $\tau \in \mathfrak{h}$. Moreover, $(u, v) \in \mathbb{Q}^2$, as the image of the poles of φ in T is not dense, but rather $(n - 1)$ -dimensional.

Note that $\mathbf{z} \mapsto \varphi(\mathbf{z}) \ominus (\mathbf{s} \cdot \mathbf{z} - u\tau - v)$ has exactly the same poles as $\mathbf{z} \mapsto \varphi(\mathbf{z})$ except for the poles which are zeros of $\mathbf{s} \cdot \mathbf{z} \equiv u\tau + v \pmod{L_\tau}$. Hence, the statement now follows inductively. By compactness of T it follows that one can restrict to only finitely many linear functions.

Remark Note that if $\mathbf{s} \cdot X \in (u, v)$ is one of the equations determining Q_φ , then $\mathbf{s} \cdot X = (u', v')$ is another equation whenever $(u, v)\gamma = (u', v')$ for some $\gamma \in \text{SL}_2(\mathbb{Z})$.

Corollary 2.10 *The vector space of strictly meromorphic Jacobi forms of some weight k , index M and poles (τ, \mathbf{z}) only in finitely many fixed hyperplanes of the form*

$$\mathbf{s} \cdot \mathbf{z} \in u\tau + v \quad (\mathbf{s} \in \mathbb{Z}^n, u, v \in \mathbb{Q}/\mathbb{Z})$$

is finite-dimensional.

Proof Recall that the ℓ th Taylor coefficient of a Jacobi form is a quasimodular form of weight $k + |\ell|$. Therefore, the multiplicity at a pole is bounded by the weight k . Now, writing $\mathbf{s}_i \cdot \mathbf{z} \in u_i\tau - v_i + L_\tau$ for the hyperplanes in the statement, indexed by $i \in I$, we find that the function $\varphi(\mathbf{z}) \prod_{i \in I} \Theta(\mathbf{s}_i \cdot \mathbf{z} - u_i\tau - v_i)^k$ is a weak Jacobi form with weight and index uniquely determined by φ and the \mathbf{s}_i . The statement now follows directly from the finite-dimensionality of the space of weak Jacobi forms of fixed weight and index.

2.3 An approximation lemma

In this section, we prove the approximation properties that were used in the proof of Proposition 2.9(ii). That is, we prove a result indicating when for given $X, Y \in M_{n,2}(\mathbb{R})$, there exists a $\gamma \in \text{SL}_2(\mathbb{Z})$ such that $X\gamma$ lies arbitrarily close to Y . For $Z \in M_{n,m}(\mathbb{R})$, write $\|Z\|$ for the distance to the closest integer matrix, i.e.

$$\|Z\| := \max_{i,j} \min_{\ell \in \mathbb{Z}} |Z_{i,j} - \ell|.$$

Definition 2.11 Given $X, Y \in M_{n,2}(\mathbb{R})$, we say Y is $\text{SL}_2(\mathbb{Z})$ -approximable by X if

$$\forall \epsilon > 0 \exists \gamma \in \text{SL}_2(\mathbb{Z}): \|X\gamma - Y\| < \epsilon.$$

We are interested in conditions on X and Y such that Y is $\text{SL}_2(\mathbb{Z})$ -approximable by X . For example, in case $n = 1$ a sufficient (but not necessary) condition is given by $X \notin M_{1,2}(\mathbb{Q})$:

Lemma 2.12 *Given $X, Y \in M_{1,2}(\mathbb{R})$ and $X \notin M_{1,2}(\mathbb{Q})$, then*

$$\forall \epsilon > 0 \exists \gamma \in \text{SL}_2(\mathbb{Z}): \|X\gamma - Y\| < \epsilon.$$

Proof Write $X = (\lambda, \mu)$. First of all, if λ/μ is irrational, then the stronger statement that the orbit of (λ, μ) under $\text{SL}_2(\mathbb{Z})$ lies dense in \mathbb{R}^2 holds (see e.g. [18]). Also, if $\mu = 0$ the orbit of (λ, μ) lies dense in $\mathbb{R}^2/\mathbb{Z}^2$ whenever λ is irrational.

From now on, we assume both λ and μ are irrational, but with rational ratio. Then,

$$(\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu \begin{pmatrix} a \frac{\lambda}{\mu} + c \\ b \frac{\lambda}{\mu} + d \end{pmatrix}.$$

Note that the matrix on the right-hand side parametrizes $N^{-1}(\mathbb{Z} \wedge \mathbb{Z})$, where N is the denominator of λ/μ and $\mathbb{Z} \wedge \mathbb{Z}$ denotes the subset of \mathbb{Z}^2 consisting of coprime integers. As μ is irrational, we conclude that the orbit of (λ, μ) under $\text{SL}_2(\mathbb{Z})$ lies dense in $\mathbb{R}^2/\mathbb{Z}^2$. Hence, if λ or μ is irrational, then $\text{SL}_2(\mathbb{Z})$ -approximability of Y by X holds.

If we replace $\text{SL}_2(\mathbb{Z})$ by $M_{n,m}(\mathbb{Z})$, there is a renowned criterion for when X approximates Y [15].

Theorem 2.13 (*Kronecker’s theorem on Diophantine approximation*) For $X, Y \in M_{n,m}(\mathbb{R})$ it holds that

$$\forall \epsilon > 0 \exists \gamma \in M_m(\mathbb{Z}): \|X\gamma - Y\| < \epsilon$$

if and only if

$$s \in \mathbb{Z}^n \text{ with } s \cdot X \in \mathbb{Z}^m \text{ implies } s \cdot Y \in \mathbb{Z}^m. \tag{12}$$

This is almost the result we are looking for, except that we want to replace $M_2(\mathbb{Z})$ by $SL_2(\mathbb{Z})$. If we were instead considering the action of SL_n on $M_{n,2}(\mathbb{R})$, the results of, for example, [17] would suffice. The action of $SL_2(\mathbb{Z})$ on $M_{1,2}(\mathbb{R})$ is dealt with in, for example, [13, 18]. The latter work already hints at the fact that the condition (12) should be altered if one replaces $M_2(\mathbb{Z})$ by $SL_2(\mathbb{Z})$. Namely, if λ, μ are coprime integers, then $(\lambda, \mu)\gamma$ is a vector of coprime integers for all $\gamma \in SL_2(\mathbb{Z})$. Hence, if $X = (\frac{1}{2}, \frac{1}{3})$, then $(0, 0)$ is not in the orbit of X for $SL_2(\mathbb{Z})$, although it is in the orbit of X for $M_2(\mathbb{Z})$. Observe that in this case the smallest $s \in \mathbb{Z}_{\geq 1}$ for which $s \cdot (0, 0) \in \mathbb{Z}^2$ does not equal the smallest $s \in \mathbb{Z}_{\geq 1}$ for which $s \cdot X \in \mathbb{Z}$ (i.e. $1 \neq 6$), which is formalized in (14).

For almost all $X = (\lambda, \mu) \in M_{n,2}(\mathbb{R})$ we have that $1, \lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n$ are linearly independent over \mathbb{Q} . In this case, and, more generally, for generic X defined below, we show one has a Diophantine approximation theorem for $SL_2(\mathbb{Z})$.

Definition 2.14 Given $X \in M_{n,2}(\mathbb{R})$, we say X is *generic* when there are $\alpha, \beta \in \mathbb{Z}$ such that for all $s \in \mathbb{Z}^n$ with $s \neq \mathbf{0}$ one has

$$s \cdot (\alpha\lambda + \beta\mu) \notin \mathbb{Z}.$$

For $s \in \mathbb{Z}^n$ and $X \in M_{n,2}(\mathbb{R})$ with $s \neq \mathbf{0}$ and $s \cdot X^t \in \mathbb{Z}^2$, we write

$$(s, s \cdot X) := \max \left\{ N \in \mathbb{Z}: \frac{s}{N} \in \mathbb{Z}^n, \frac{s \cdot X}{N} \in \mathbb{Z}^2 \right\}$$

for the greatest common divisor of the entries of s and $s \cdot X$.

Proposition 2.15 (*Partial result on Diophantine approximation for $SL_2(\mathbb{Z})$*) Let $X, Y \in M_{n,2}(\mathbb{R})$. Then, if X is generic one has

$$\forall \epsilon > 0 \exists \gamma \in SL_2(\mathbb{Z}): \|X\gamma - Y\| < \epsilon. \tag{13}$$

Conversely, if (13) holds, then for all non-trivial $s \in \mathbb{Z}^n$ for which $s \cdot X \in \mathbb{Z}^2$ one has that $s \cdot Y \in \mathbb{Z}^2$ and

$$(s, s \cdot X) = (s, s \cdot Y). \tag{14}$$

Proof Suppose that $s \in \mathbb{Z}^n$ is such that $s \cdot X \in \mathbb{Z}^2$ (as in the second part of the statement). Observe that $s \cdot (X\gamma) \in \mathbb{Z}^2$ for all $\gamma \in SL_2(\mathbb{Z})$. Hence, for any $\gamma \in SL_2(\mathbb{Z})$ one has

$$\|X\gamma - Y\| \geq \frac{1}{n} \|s \cdot (X\gamma) - s \cdot Y\| = \frac{1}{n} \|s \cdot Y\|.$$

Therefore, if (13) holds, then for all $s \in \mathbb{Z}^n$ with $s \cdot X \in \mathbb{Z}^2$ one has $s \cdot Y \in \mathbb{Z}^2$.

Next, suppose $\mathbf{s} \in \mathbb{Z}^n$ such that $\mathbf{s} \cdot X, \mathbf{s} \cdot Y \in \mathbb{Z}^2$ and $\mathbf{s} \neq \mathbf{0}$. Write $N = (\mathbf{s}, \mathbf{s} \cdot Y)$ and $\mathbf{s}' = N^{-1}\mathbf{s}$. Suppose $\mathbf{s}' \cdot (X\gamma) \notin \mathbb{Z}^2$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$, i.e. $\mathbf{s}' \cdot (X\gamma) \in (N^{-1}\mathbb{Z}^2) \setminus \mathbb{Z}^2$. Therefore, if (13) holds, for any $\gamma \in \text{SL}_2(\mathbb{Z})$ one has

$$\|X\gamma - Y\| \geq \frac{1}{n} \|\mathbf{s}' \cdot (X\gamma) - \mathbf{s}' \cdot Y\| = \frac{1}{n} \|\mathbf{s}' \cdot (X\gamma)\| \geq \frac{1}{Nn},$$

which is a contradiction. Therefore, we have that $\mathbf{s}' \cdot (X\gamma) \in \mathbb{Z}^2$ for some $\gamma \in \text{SL}_2(\mathbb{Z})$. This implies that $\mathbf{s}' \cdot X \in \mathbb{Z}^2$. Hence, $(\mathbf{s}, \mathbf{s} \cdot Y) \leq (\mathbf{s}, \mathbf{s} \cdot X)$. Analogously, the other inequality $(\mathbf{s}, \mathbf{s} \cdot X) \leq (\mathbf{s}, \mathbf{s} \cdot Y)$ holds. We conclude that $(\mathbf{s}, \mathbf{s} \cdot X) = (\mathbf{s}, \mathbf{s} \cdot Y)$.

Next, let $X = (\lambda, \mu)$ be generic, i.e. let $\alpha, \beta \in \mathbb{Z}$ be such that for all non-trivial $\mathbf{s} \in \mathbb{Z}^n$ we have $\mathbf{s} \cdot (\alpha\lambda + \beta\mu) \notin \mathbb{Z}$. Assume without loss of generality that α and β are coprime. Choose $\gamma_0 \in \text{SL}_2(\mathbb{Z})$ such that its first column is given by $(\alpha, \beta)^t$ and write $X\gamma_0 = (\tilde{\lambda}, \tilde{\mu})$. By construction, $\tilde{\lambda} = \alpha\lambda + \beta\mu$. Hence, by Kronecker's theorem we find infinitely many $b \in \mathbb{Z}$ such that

$$\|b\tilde{\lambda} + \tilde{\mu} - \xi\| < \epsilon, \tag{15}$$

where we denoted $Y = (\nu, \xi)$. Suppose for all such b there exists a non-trivial $\mathbf{s}(b) \in \mathbb{Z}^n$ such that $\mathbf{s}(b) \cdot (b\tilde{\lambda} + \tilde{\mu}) \in \mathbb{Z}$. Note that there exist finitely many integers a_b with $\sum_b a_b \mathbf{s}(b) = \mathbf{0}$ and $\sum_b b a_b \mathbf{s}(b) \neq \mathbf{0}$. Hence, we find

$$\mathbb{Z} \ni \sum_b a_b \mathbf{s}(b) \cdot (b\tilde{\lambda} + \tilde{\mu}) = \sum_b b a_b \mathbf{s}(b) \cdot \tilde{\lambda},$$

contradicting our assumption that X is generic. Hence, for a suitably chosen $b \in \mathbb{Z}$ both (15) holds and there exists a $c \in \mathbb{Z}$ such that

$$\|c(b\tilde{\lambda} + \tilde{\mu}) + \tilde{\lambda} - \nu\| < \epsilon.$$

In other words, for $\gamma = \begin{pmatrix} bc+1 & b \\ c & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, we have

$$\|X\gamma_0\gamma - Y\| < \epsilon,$$

as desired.

Remark Condition (14) is necessary for $\text{SL}_2(\mathbb{Z})$ -approximability of Y by X . Whether this condition also suffices, or, if not, how it should be strengthened to a necessary and sufficient condition remains an open problem.

2.4 Quasi-Jacobi forms

Consider the real-analytic functions $\nu : \mathfrak{h} \rightarrow \mathbb{C}$ and $\xi : \mathfrak{h} \times \mathbb{C} \rightarrow \mathbb{C}$, given by

$$\nu(\tau) := \frac{1}{2i \text{Im}(\tau)}, \quad \xi(\tau, z) := \frac{\text{Im}(z)}{\text{Im}(\tau)}.$$

These functions almost transform as a Jacobi form:

$$\begin{aligned} (\nu|_{2,0}\gamma)(\tau) &= \nu(\tau) - \frac{c}{c\tau + d} & (\xi|_{1,0}\gamma)(\tau, z) &= \xi(\tau, z) - \frac{cz}{c\tau + d} \\ (\nu|_0X)(\tau) &= \nu(\tau) & (\xi|_0X)(z) &= \xi(z) + \lambda. \end{aligned}$$

Definition 2.16 Let $k \in \mathbb{Z}, M \in M_n(\mathbb{Q})$. Denote by C_n a subspace of all strictly meromorphic functions $\mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$. An almost Jacobi form Φ of rank n , weight k , index M and analytic type C_n satisfies:

- (i) $\Phi \in C_n[v(\tau), \xi(\tau, z_1), \dots, \xi(\tau, z_n)]$;
- (ii) $\Phi|_{k,M} g = \Phi$ for all $g \in \Gamma_n^J$.

Definition 2.17 A *quasi-Jacobi form* φ is the constant term with respect to v and ξ of an almost Jacobi form. If C_n equals Hol_n^M , Hol_n or Mer_n^M , φ is a *holomorphic, weak* or *strictly meromorphic quasi-Jacobi form*, respectively. We write $\tilde{\mathcal{J}}^{\text{hol}}$, $\tilde{\mathcal{J}}^{\text{weak}}$ and $\tilde{\mathcal{J}}^{\text{sm}}$ for the algebras of holomorphic, weak, and strictly meromorphic quasi-Jacobi forms.

As a first example, quasimodular forms are quasi-Jacobi forms of rank $n = 0$. More interestingly, the functions

$$E_2(\tau, z) - 2\pi i v(\tau), \quad E_1(\tau, z) + 2\pi i \xi(\tau, z),$$

are almost strictly meromorphic Jacobi forms of index 0 and weight 2 and 1, respectively. Hence, E_2 and E_1 are strictly meromorphic quasi-Jacobi forms. Observe that

$$E_2(\tau, z) = -D_z E_1(\tau, z) \quad E_1(\tau, z) = (2\pi i) D_z \log \Theta(\tau, z) \quad \left(D_z = \frac{1}{2\pi i} \frac{\partial}{\partial z} \right),$$

which reminds one of

$$e_2 = 8\pi^2 D_\tau \log \eta, \quad \left(\eta(\tau) = q^{1/24} \prod_n (1 - q^n), \quad D_\tau = \frac{1}{2\pi i} \frac{\partial}{\partial \tau} = q \frac{\partial}{\partial q} \right).$$

Remark Let Φ be the meromorphic almost Jacobi form corresponding to φ and write

$$\Phi(\tau, z) = \sum_{i,j} \varphi_{i,j}(\tau, z) v(\tau)^i \xi(\tau, z_1)^{j_1} \dots \xi(\tau, z_n)^{j_n}.$$

Making use of the algebraic independence of v and ξ over the field of meromorphic functions, we find $\varphi \in \tilde{\mathcal{J}}^{\text{sm}}$ precisely if $\varphi \in \text{Mer}_n^M$ and there exist a finite number of $\varphi_{i,j} \in \text{Mer}_n^M$, indexed by a subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n$, satisfying

$$\begin{aligned} (\varphi|\gamma)(\tau, z) &= \sum_{i,j} \varphi_{i,j}(\tau, z) \left(\frac{c}{c\tau + d} \right)^{i+|j|} \frac{z^j}{(2\pi i)^i}; \\ (\varphi|X)(z) &= \sum_j \varphi_{0,j}(z) (-\lambda)^j. \end{aligned}$$

These are equations (8) and (9) in the Introduction. (Recall that for vectors $\mathbf{a} \in \mathbb{C}^n$, $\mathbf{b} \in \mathbb{Z}^n$ we write $\mathbf{a}^{\mathbf{b}} = \prod_r a_r^{b_r}$.)

The strictly meromorphic quasi-Jacobi forms e_2 and E_1 play a central role as building blocks of quasi-Jacobi forms out of Jacobi forms. For convenience, we introduce the following alternative normalizations

$$\mathfrak{e}_2 := \frac{1}{4\pi^2} e_2 = \frac{1}{12} - 2 \sum_{m,r \geq 1} m q^{mr}, \quad A = \frac{1}{2\pi i} E_1.$$

Quasi-Jacobi forms are not invariant under the action of the Jacobi group. However, the fact that an almost Jacobi form is a polynomial in v and ξ implies that a quasi-Jacobi form transforms “up to a polynomial correction” as a Jacobi form, or, equivalently, that it is a polynomial in the strictly meromorphic quasi-Jacobi forms $\mathfrak{e}_2(\tau)$ and $A(\tau, z)$ with Jacobi forms as coefficients.

Proposition 2.18 *An equivalent definition for a strictly meromorphic quasi-Jacobi form is as follows: $\varphi \in \tilde{J}_{k,M}^{\text{sm}}$ if $\varphi \in \text{Mer}_n^M$ and there exist a finite number of $\psi_{i,j} \in J_{k-2i-|j|,M}^{\text{sm}}$ indexed by a subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n$, such that*

$$\varphi(\mathbf{z}) = \sum_{i,j} \psi_{i,j}(\mathbf{z}) \mathfrak{e}_2^i A(z_1)^{j_1} \cdots A(z_n)^{j_n}. \tag{16}$$

Proof We define the coefficients $\psi_{i,j}$ by the expansion of the almost Jacobi form Φ corresponding to φ , i.e.

$$\Phi(\tau, \mathbf{z}) =: \sum_{i,j} \psi_{i,j}(\tau, \mathbf{z}) \left(\mathfrak{e}_2(\tau) + \frac{\nu(\tau)}{2\pi i} \right)^i \prod_r (A(\tau, z_r) + \xi(\tau, z_r))^{j_r}. \tag{17}$$

Note that $\mathfrak{e}_2(\tau) + \frac{\nu(\tau)}{2\pi i}$ and $A(\tau, z_r) + \xi(\tau, z_r)$ transform as Jacobi forms. Moreover, they are algebraically independent over the space of all meromorphic functions. Hence, it follows that the coefficients $\psi_{i,j}$ are strictly meromorphic Jacobi forms. The constant term with respect to i and j , by definition equal to φ , is now easily seen to be equal to the right-hand side of (16).

Remark From (17) it follows that the $\varphi_{i,j}$ are quasi-Jacobi forms related to the $\psi_{i,j}$ by

$$\varphi_{i,j}(\mathbf{z}) = \sum_{i',j'} \binom{i+i'}{i} \binom{j+j'}{j} \psi_{i+i',j+j'}(\mathbf{z}) \mathfrak{e}_2^{i'} \prod_r A(z_r)^{j'_r},$$

where

$$\binom{j+j'}{j} := \prod_r \binom{j_r+j'_r}{j_r}.$$

Also, note that given a representation for φ as in (16) one has

$$\varphi_{1,0} = \frac{\partial}{\partial \mathfrak{e}_2} \varphi \quad \text{and} \quad \varphi_{0,e_i}(\mathbf{z}) = \frac{\partial}{\partial A(z_i)} \varphi(\mathbf{z}).$$

By (16) quasi-Jacobi forms share the properties of Jacobi forms with respect to the location of the poles:

Corollary 2.19 *The statement of Proposition 2.9 also holds when φ is a strictly meromorphic quasi-Jacobi form.*

As a corollary of Theorem 2.7, we have the following representations for the algebras J^0 and \tilde{J}^0 of all rank one strictly meromorphic Jacobi forms and strictly meromorphic quasi-Jacobi forms with all poles at the lattice points L_τ :

$$J^0 = \mathbb{C}[E_2 - e_2, E_3, e_4, \Theta], \quad \tilde{J}^0 = \mathbb{C}[E_1, E_2, E_3, e_2, e_4, \Theta].$$

Observe the vector subspaces for fixed weight and index are finite-dimensional. This holds more generally.

Corollary 2.20 *The space of all meromorphic quasi-Jacobi forms of some weight k , index M , and with all poles in a finite union of fixed rational hyperplanes as in Theorem 1.6 is finite-dimensional.*

Proof This follows directly from the previous proposition as by Corollary 2.10 the number of linearly independent $\psi_{i,j}$ is finite.

The operators $D_\tau = \frac{1}{2\pi i} \frac{\partial}{\partial \tau}$ and $D_{z_i} = \frac{1}{2\pi i} \frac{\partial}{\partial z_i}$ preserve the space of quasi-Jacobi forms ($1 \leq i \leq n$). This leads to yet another equivalent definition of quasi-Jacobi forms, as derivatives of Jacobi forms. Note that no power of the quasi-Jacobi form e_2 (which is in fact a quasimodular form and hence of trivial index) can be written in terms of derivatives of Jacobi forms. However, in case the index is positive definite, by [20, Proposition 1(i)] we have the following.

Proposition 2.21 *Let φ be a quasi-Jacobi form of weight k and positive definite index M . Then, there exist unique Jacobi forms $\psi_{\mathbf{d}}$ with $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{n+1}$ of weight $k - 2d_0 - d_1 - \dots - d_n$ and index M such that*

$$\varphi = \sum_{\mathbf{d}} D_\tau^{d_0} D_{z_1}^{d_1} \dots D_{z_n}^{d_n} \psi_{\mathbf{d}}.$$

Proof Choose an ordering on \mathbb{Z}^{n+1} respecting the ordering on \mathbb{Z} . Given a Jacobi form φ , let (i, \mathbf{j}) be maximal (with respect to this ordering) for which $\varphi_{i,j}$ in Proposition 2.18 exists and is nonzero. A direct check using the same proposition shows that φ minus a multiple of $D_\tau^{j_0} D_{z_1}^{j_1} \dots D_{z_n}^{j_n} \varphi_{i,j}$ is a quasi-Jacobi form for which this maximal index is smaller. Here, as M is positive definite, this multiple is nonzero.

2.5 Action of the Jacobi Lie algebra by derivations

The last proposition depended on the derivations D_τ and D_{z_i} . Here, we study natural derivations on the space of quasi-Jacobi forms. Given a quasi-Jacobi form φ , recall the quasi-Jacobi forms $\varphi_{i,j}$ (see Equations (8) and (9) or the previous section) are defined for (i, \mathbf{j}) in a finite index set I with $I \subset \mathbb{Z} \times \mathbb{Z}^n$. By convention, we let $\varphi_{i,j} := 0$ if $(i, \mathbf{j}) \notin I$.

Definition 2.22 Let δ_τ and δ_{z_i} be the derivations on the space of quasi-Jacobi forms given by $\varphi \mapsto \varphi_{1,0}$ and $\varphi \mapsto \varphi_{0,e_i}$, respectively (with e_i the standard i th basis vector of \mathbb{R}^n).

Observe that the functions $\varphi_{i,j}$ are given by

$$\varphi_{i,j} = \frac{\delta_\tau^i \delta_{\mathbf{z}}^{\mathbf{j}}}{i! j!} \varphi \qquad (\delta_{\mathbf{z}}^{\mathbf{j}} = \delta_{z_1}^{j_1} \dots \delta_{z_n}^{j_n}, \mathbf{j}! = j_1! \dots j_n!).$$

Now, a *key observation* for the rest of this work is that the transformation behaviour of φ is uniquely determined by the action of the operators $\delta_\tau^i \delta_{\mathbf{z}}^{\mathbf{j}}$ on φ . In particular, the transformation behaviour of the Taylor coefficients of φ —which are quasimodular forms—is determined by the action of δ_τ^i on these coefficients. In the next section, we investigate how the transformation behaviour of a quasi-Jacobi form determines the transformation of its Taylor coefficients, and vice versa, by studying the action of these derivations.

Remark Writing φ as in (16) yields

$$\delta_\tau \varphi = \frac{\partial}{\partial \mathfrak{e}_2} \varphi, \quad (\delta_{z_i} \varphi)(\mathbf{z}) = \frac{\partial}{\partial A(z_i)} \varphi(\mathbf{z}).$$

The operators $D_\tau, D_{z_i}, \delta_\tau$ and δ_{z_i} are part of a Lie algebra of operators acting on quasi-Jacobi forms by derivations, as we explain now. Following a suggestion by Zagier, we consider the notion of a \mathfrak{g} -algebra for any Lie algebra \mathfrak{g} , defined as follows.

Definition 2.23 Given a Lie algebra \mathfrak{g} , a \mathfrak{g} -algebra is an algebra A together with a Lie homomorphism $\mathfrak{g} \rightarrow \text{Der}(A)$, where $\text{Der}(A)$ denotes the Lie algebra of all derivations on A .

Denote by W and I_{ij} the weight and index operators acting diagonally by multiplying with the weight k and $B_M(e_i, e_j)$, respectively, where B_M is the bilinear form corresponding to the index M . Let \mathfrak{j} be the Lie algebra of the Jacobi group. By [20, Eqn. (12)] the Lie algebra of the Jacobi group acts by the aforementioned derivations on the space of quasi-Jacobi forms.

Proposition 2.24 *The algebra of quasi-Jacobi forms is a \mathfrak{j} -algebra, i.e. the algebra of derivations $D_\tau, D_{z_i}, \delta_\tau, \delta_{z_i}, W$ and I_{ij} is isomorphic to \mathfrak{j} and acts on the space of quasi-Jacobi forms.*

Remark More concretely, the commutation relations of (i) the modular operators, (ii) the elliptic operators and (iii) their interactions are given by

$$\begin{aligned} \text{(i)} \quad & [\delta_\tau, D_\tau] = W, & [W, D_\tau] = 2D_\tau, & [W, \delta_\tau] = -2\delta_\tau, \\ \text{(ii)} \quad & [\delta_{z_i}, D_{z_j}] = 2I_{ij}, & [I_{ij}, D_{z_i}] = 0, & [I_{ij}, \delta_{z_i}] = 0, \\ \text{(iii)} \quad & [\delta_{z_i}, D_\tau] = D_{z_i}, & [\delta_\tau, D_{z_i}] = \delta_{z_i}, & [W, D_{z_i}] = D_{z_i}. \end{aligned}$$

The other commutators vanish. As the spaces of almost Jacobi forms and quasi-Jacobi forms are isomorphic, the same result holds for almost Jacobi forms when one replaces δ_τ by $2\pi i \frac{\partial}{\partial v}$ and δ_z by $\frac{\partial}{\partial \xi(z)}$.

2.6 The double slash operator

A holomorphic Jacobi form has two important representations: the *theta expansion* and the *Taylor expansion*. We generalize the Taylor expansion to strictly meromorphic quasi-Jacobi forms in such a way that the Taylor coefficients are quasimodular forms. Moreover, we give criteria based on the coefficients in these representations for a meromorphic function to be a quasi-Jacobi form.

Given a Jacobi form φ and $X \in M_{n,2}(\mathbb{Q})$, the Taylor coefficients of $(\varphi|X)(\mathbf{z})$ around $\mathbf{z} = \mathbf{0}$ are quasimodular forms for the group

$$\Gamma_X = \{\gamma \in \text{SL}_2(\mathbb{Z}) \mid X\gamma - X \in M_{n,2}(\mathbb{Z}), \rho(X - X\gamma) = \zeta_{X, X\gamma - X}\}, \quad (18)$$

where ρ and $\zeta_{X, X'}$ are defined by (10). In contrast to Jacobi forms, it is not true that the Taylor coefficients of quasi-Jacobi forms are quasimodular. Namely, as stated the

Introduction, for $X = (\lambda, \mu) \in M_{n,2}(\mathbb{Q})$ one has that $(\Theta'|X|\gamma)(\tau, z)$ equals—up to the multiplicative constant $\rho(X)\rho(-X\gamma)$ —

$$(\Theta'|X\gamma)(\tau, z) + \frac{cz}{c\tau + d}(\Theta|X\gamma)(\tau, z) + \lambda(\Theta|X\gamma)(\tau, z) - \frac{\lambda}{c\tau + d}(\Theta|X\gamma)(\tau, z), \tag{19}$$

for all $\gamma \in \Gamma_X$. All but the last term $-\frac{\lambda}{c\tau+d}(\Theta|X)(\tau, z)$ of (19) depend polynomially on $\frac{c}{c\tau+d}$, so that the Taylor coefficients of $(\Theta'|X)(\tau, z)$ at $z = \mathbf{0}$ are not transforming in accordance with the quasimodular transformation formula. Note that this last term can be written as $-\lambda(\Theta|X|_0\gamma)(\tau, z)$. Here, it should be noted that the weight 0 in the slash operator is unusual. Namely, Θ is of weight -1 , whereas Θ is of weight 0. In conclusion, the function

$$\Theta' \| X := \rho(-X) (\Theta'|X + \lambda \Theta|X)$$

rather than $\Theta'|X$ transforms as a quasi-Jacobi form of weight 0, i.e. for $\gamma \in \Gamma_X$ one has

$$(\Theta' \| X|_0\gamma)(\tau, z) = (\Theta' \| X)(\tau, z) + \frac{cz}{c\tau + d}(\Theta|X)(\tau, z).$$

We now introduce the double slash action $(\varphi \| X)$ for any quasi-Jacobi form φ and all $X \in M_{n,2}(\mathbb{R})$. In the next section, we will use this notation to define the Taylor coefficients of φ at X .

Definition 2.25 Given $M \in M_n(\mathbb{Q})$ and a family of functions $\varphi_{0,j} : \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$ indexed by a finite subset of $\mathbb{Z}_{\geq 0}^n$ (with $\varphi := \varphi_{0,0}$), define the *double slash operator* by

$$\varphi \|_M X := \rho(-X) \sum_j (\varphi_{0,j} |_M X) \mathcal{X}^j,$$

where ρ is given by (10).

Convention 2.26 In case φ is a quasi-Jacobi form, in this definition, we always take the family $\varphi_{0,j}$ determined by the elliptic transformation (16).

Proposition 2.27 Given a family of functions $\varphi_{i,j} : \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$ indexed by a finite subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n$ (with $\varphi = \varphi_{0,0}$) and $X \in M_{n,2}(\mathbb{R})$, one has

(i) If φ satisfies the quasimodular transformation (8) for Γ , then

$$(\varphi \| X|\gamma)(\tau, z) = \sum_{i,j} (\varphi_{i,j} \| X\gamma)(\tau, z) \left(\frac{c}{c\tau + d}\right)^{i+|j|} \frac{z^j}{(2\pi i)^i} = (\varphi|\gamma \| X\gamma)(\tau, z)$$

for all $\gamma \in \Gamma$.

(ii) If φ satisfies the quasi-elliptic transformation (9), then

$$\zeta_{X,X'} \varphi \| X \| X' = \zeta_{X,X'} \varphi \| X' \| X = \rho(-X') \zeta_{X,X'} \varphi \| X = \varphi \| (X + X')$$

for all $X' \in M_{n,2}(\mathbb{Z})$, where the root of unity $\zeta_{X,X'}$ is defined by (10).

(iii) If φ is a quasi-Jacobi form for $SL_2(\mathbb{Z})$, then $\varphi \| X$ is a quasi-Jacobi form for Γ_X , and

$$\varphi \| (X + X') = \rho(-X') \zeta_{X,X'} \varphi \| X$$

for all $X' \in M_{n,2}(\mathbb{Z})$.

Proof The transformation of $\varphi \|X$ under the Jacobi group follows by direct computations. We often make use of

$$\frac{\delta_{\tau}^{i'} \delta_z^{j'}}{i'! j'!} \varphi_{i,j} = \frac{\delta_{\tau}^{i'} \delta_z^{j'}}{i'! j'!} \frac{\delta_{\tau}^i \delta_z^j}{i! j!} \varphi = \binom{i+i'}{i} \binom{j+j'}{j} \varphi_{i+i',j+j'},$$

where (as before)

$$\binom{j+j'}{j} = \prod_r \binom{j_r+j'_r}{j_r}.$$

The first property follows from the following:

$$\begin{aligned} \varphi \|X\|_k \gamma &= \rho(-X) \sum_{\ell} (\varphi_{0,\ell} |X|_k \gamma) \lambda^{\ell} \\ &= \rho(-X\gamma) \sum_{\ell} (\varphi_{0,\ell} |k\gamma|X\gamma) \lambda^{\ell} \\ &= \rho(-X\gamma) \sum_{i,j,\ell} (\varphi_{i,j+\ell} |X\gamma) \left(\frac{1}{2\pi i} \frac{c}{c\tau+d} \right)^i \binom{j+\ell}{j} \left(\frac{c(z+\lambda^{\gamma}\tau+\mu^{\gamma})}{c\tau+d} \right)^j \left(\frac{\lambda}{c\tau+d} \right)^{\ell} \\ &= \rho(-X\gamma) \sum_{i,j} (\varphi_{i,j} |X\gamma) \left(\frac{1}{2\pi i} \frac{c}{c\tau+d} \right)^i \left(\frac{c(z+\lambda^{\gamma}\tau+\mu^{\gamma})+\lambda}{c\tau+d} \right)^j \\ &= \rho(-X\gamma) \sum_{i,j} (\varphi_{i,j} |X\gamma) \left(\frac{1}{2\pi i} \frac{c}{c\tau+d} \right)^i \left(\frac{cz}{c\tau+d} + \lambda^{\gamma} \right)^j \\ &= \rho(-X\gamma) \sum_{i,j,\ell} (\varphi_{i,j+\ell} |X\gamma) \left(\frac{1}{2\pi i} \frac{c}{c\tau+d} \right)^i \binom{j+\ell}{j} \left(\frac{cz}{c\tau+d} \right)^j (\lambda^{\gamma})^{\ell} \\ &= \sum_{i,j} (\varphi_{i,j} \|X\gamma) \left(\frac{1}{2\pi i} \frac{c}{c\tau+d} \right)^i \left(\frac{cz}{c\tau+d} \right)^j. \end{aligned}$$

For the second property, observe that

$$\begin{aligned} \varphi \|X\|X' &= \rho(-X') \sum_{\ell} (\varphi_{0,\ell} \|X|X') (\lambda')^{\ell} \\ &= \rho(-X)\rho(-X') \sum_{j,\ell} (\varphi_{0,j+\ell} |X|X') \binom{j+\ell}{j} \lambda^j (\lambda')^{\ell}, \end{aligned}$$

from which it is clear that $\zeta_{X',X} \varphi \|X\|X' = \zeta_{X,X'} \varphi \|X'|X$. Moreover, by the elliptic transformation

$$\begin{aligned} \varphi \|X'|X &= \rho(-X)\rho(-X') \sum_{j,\ell,m} (\varphi_{0,j+\ell+m} |X) \binom{j+\ell+m}{j,\ell,m} \lambda^j (\lambda')^{\ell} (-\lambda')^m \\ &= \rho(-X)\rho(-X') \sum_j (\varphi_{0,j} |X) \lambda^j \\ &= \rho(-X') \varphi \|X. \end{aligned}$$

Next, one has

$$\begin{aligned}
 (\varphi \| X + X') &= \rho(-X - X') \sum_{\ell} (\varphi_{0,\ell} | X + X') (\lambda + \lambda')^{\ell} \\
 &= \rho(-X) \rho(-X') \zeta_{X,X'} \sum_{\ell} (\varphi_{0,\ell} | X' | X) (\lambda + \lambda')^{\ell} \\
 &= \rho(-X) \rho(-X') \zeta_{X,X'} \sum_{j,\ell} (\varphi_{0,j+\ell} | X) \binom{j+\ell}{j} (-\lambda')^j (\lambda + \lambda')^{\ell} \\
 &= \rho(-X) \rho(-X') \zeta_{X,X'} \sum_j (\varphi_{0,j} | X) \lambda^j \\
 &= \rho(-X') \zeta_{X,X'} \varphi \| X.
 \end{aligned}$$

Finally, the fact that $\varphi \| X$ is a quasi-Jacobi form follows directly from the definition of Γ_X and the previous properties.

2.7 Taylor coefficients

Let $X = (\lambda, \mu) \in M_{n,2}(\mathbb{Q}), M \in M_n(\mathbb{Q})$ and $\varphi \in \text{Mer}_n$. We now study the Taylor coefficients of $\varphi \|_M X$ around $\mathbf{z} = \mathbf{0}$. In case φ is a strictly meromorphic quasi-Jacobi form, recall that all poles \mathbf{z} lie on hyperplanes of the form $\mathbf{s} \cdot \mathbf{z} \in u\tau + v$ for some $\mathbf{s} \in \mathbb{Z}^n$ and $u, v \in \mathbb{Q}/\mathbb{Z}$ by Theorem 1.6. From now on, we assume that $\mathbf{s} = \mathbf{e}_i$ for some i so that a Laurent series of $\varphi \| X$ of the form

$$\sum_{\ell_1 \geq L} \cdots \sum_{\ell_n \geq L} a_{\ell_1, \dots, \ell_n} (z_1 - \lambda_1 \tau - \mu_1)^{\ell_1} \cdots (z_n - \lambda_n \tau - \mu_n)^{\ell_n}$$

for some $L \in \mathbb{Z}$ and $a_{\ell} \in \mathbb{C}$ exists. For example, the poles of all the meromorphic quasi-Jacobi forms we encounter in the applications lie on the coordinate axes.

Definition 2.28 We call the poles of a meromorphic function $\varphi: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$ *orthogonal* if the set of poles of $\varphi(\tau, \cdot)$ is given by a union of special hyperplanes of the form

$$z_j \in u\tau + v$$

for some $j \in \{1, \dots, n\}$ and $u, v \in \mathbb{Q}/\mathbb{Z}$.

Making use of the notions of *orthogonal* poles (defined above) and the double slash action (defined in Definition 2.25), we will now define the ‘‘Taylor coefficients’’ of a family of functions in the following way. Recall that in case φ is a Jacobi form, there is a canonical choice for the family of functions φ which is part of the data of these ‘‘Taylor coefficients’’ (see Convention 2.26).

Definition 2.29 Let $M \in M_n(\mathbb{Q})$ and $\varphi = \{\varphi_{i,j}\}$, where $\varphi_{i,j}: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$ is a family of meromorphic functions indexed by a finite subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n$, with $\varphi := \varphi_{0,0} \in \text{Mer}_n^M$ such that all poles of φ are orthogonal. Define $g_{\ell}(\varphi)$ as the ℓ th Laurent coefficient of φ :

$$\varphi(\tau, \mathbf{z}) =: \sum_{\ell} g_{\ell}(\varphi)(\tau) \mathbf{z}^{\ell}.$$

For all $X \in M_{n,2}(\mathbb{R})$, we define the ℓ th ‘‘Taylor coefficient’’ $g_{\ell}^X(\varphi)$ of φ as $g_{\ell}(\varphi \|_M X)$.

Also, denote

$$g_{\ell,s}^{X,r}(\varphi) = g_{\ell} \left(B_M^r(\mathbf{z}, \mathbf{z}) \sum_{i+|j|=s} (\varphi_{i,j} \| X)(\mathbf{z}) \frac{\mathbf{z}^j}{(2\pi i)^i} \right) \quad (r, s \in \mathbb{Z}_{\geq 0}).$$

We often omit r, s or X from the notation if $r = 0, s = 0$ or $X = (\mathbf{0}, \mathbf{0})$.

Remark • The functions $g_{\ell,s}^{X,r}(\varphi)$ naturally appear if one studies the action of δ_τ on the “Taylor coefficients” $g_\ell^X(\varphi)$ of φ .

- $g_{\ell,s}^r(\varphi) = \frac{1}{s!} g_\ell(B_M(\mathbf{z}, \mathbf{z})^r (\delta_\tau + z_1 \delta_{z_1} + \dots + z_n \delta_{z_n})^s \varphi)$.
- In case $r = 0$, one may be tempted to write

$$g_{\ell,s}^{X,0}(\varphi) = \sum_{i+|j|=s} \frac{1}{(2\pi i)^i} g_{\ell-j}(\varphi_{i,j} \| X)''.$$

However, we do not assume that the functions $\varphi_{i,j}$ admit orthogonal poles, so the Taylor expansion of $\varphi_{i,j}$ may not exist. For example, taking $\varphi = F_2$ (defined by Definition 1.1), we will see later that $\varphi_{0,e_1}(z_1, z_2) = \frac{1}{\Theta(z_1+z_2)}$, which has a pole whenever $z_1 + z_2 = 0$. Theorem 2.33 implicitly shows that the notation $g_{\ell,s}^{X,r}(\varphi)$ is well-defined for a quasi-Jacobi form φ with all poles orthogonal.

The data $\{g_\ell^X(\varphi)\}$ uniquely determine φ as well as the family $\varphi = \{\varphi_{i,j}\}$. Hence, it is natural to ask under which conditions on $g_\ell^X(\varphi)$ the function φ is a meromorphic quasi-Jacobi form. Before we answer this question, we study the modular properties of $g_\ell^X(\varphi)$ given φ is a quasi-Jacobi form. As a corollary of the previous proposition on $\varphi \| X$, generalizing [9, Theorem 1.3] to quasi-Jacobi forms, we first show that $g_0^X(\varphi)$ is a quasimodular form in case φ is holomorphic.

Corollary 2.30 *Let φ be a holomorphic quasi-Jacobi form of weight k and index M . For all $X \in M_{n,2}(\mathbb{Q})$, the function $g_0^X(\varphi)$ is a holomorphic quasimodular form of weight k for the group Γ_X (defined by (18)). Moreover,*

$$\delta_\tau g_0^X(\varphi) = g_0^X(\delta_\tau \varphi).$$

Proof Let $X \in M_{n,2}(\mathbb{Q})$ and $\gamma \in \Gamma_X$. Then, by Proposition 2.27(2.27) one finds

$$(g_0^X(\varphi) | \gamma)(\tau) = (\varphi \| X | \gamma)(\tau, \mathbf{0}) = \sum_i g_0^X(\varphi_i) \left(\frac{1}{2\pi i} \frac{c}{c\tau + d} \right)^i,$$

where φ_i denotes the family corresponding to $\varphi_{i,0}$. Holomorphicity in \mathfrak{h} and at infinity follows directly as φ is a holomorphic Jacobi form. Hence, $g_0^X(\varphi)$ is a quasimodular form for this group and $\delta_\tau^r g_0^X(\varphi) = \frac{1}{r!} g_0^X(\varphi_r) = g_0^X(\delta_\tau^r \varphi)$.

In the above corollary, observe that if φ is a true Jacobi form (rather than a quasi-Jacobi form), then $\delta_\tau \varphi = 0$. Correspondingly, in that case $\delta_\tau g_0^X(\varphi) = 0$ and $g_0^X(\varphi)$ is a true modular form (rather than a quasimodular form).

The quasimodularity of the other coefficients $g_\ell^X(\varphi)$ can be understood in terms of lower coefficients in two ways. First of all, certain linear combinations of derivatives of these coefficients are modular. Secondly the action of δ_τ^i on $g_\ell^X(\varphi)$ can be expressed in terms of other coefficients.

We first show that these two ways are equivalent. Recall that $(x)_n$ denotes the Pochhammer symbol $(x)_n = x(x+1) \cdots (x+n-1)$.

Proposition 2.31 *Let $g = g_k, g_{k-2}, \dots, g_{k-2p}$ be quasimodular forms of depth at most p and weight $k, k-2, \dots, k-2p$, respectively. Then, the following are equivalent:*

- (i) $\delta_\tau^i g = g_{k-2i}$ for $i = 0, \dots, p$;
- (ii) The functions

$$\begin{cases} \sum_{0 \leq m \leq p-i} (-1)^m \frac{D_\tau^m g_{k-2i-2m}}{(k-2i-m-1)_m m!} & \text{if } k > 2p \text{ or } i < p-1, \\ g_2 - \mathbb{e}_2 g_0 & \text{if } k = 2p \text{ and } i = p-1 \end{cases}$$

for $i = 0, \dots, p-1$ are modular forms of weight $k-2i$;

- (iii) The functions

$$\sum_{0 \leq m \leq p-i} (-1)^m \frac{(D_\tau + \mathbb{e}_2)^m g_{k-2i-2m}}{(k-2i-m-\frac{3}{2})_m m!}$$

for $i = 0, \dots, p-1$ are modular forms of weight $k-2i$.

Remark Observe that if $m \leq p-i$ and $i \leq p-1$ one has that

$$k-2i-m-1 \geq k-2p.$$

Let $k = 2p$, $i = p-1$ and take $m = p-i$, i.e. $m = 1$. Then, the numerator $(k-2i-m-1)_m$ vanishes. Moreover, for these values of k, i and m , the function $g_{k-2i-2m}$ is a modular form of weight 0. Hence, it is a constant function. Therefore, also the numerator $D_\tau^m g_{k-2i-2m}$ vanishes in this case. One can think of \mathbb{e}_2 as being the appropriate regularization of the corresponding ill-defined ratio.

Observe that in the third equivalence we replaced D_τ by $D_\tau + \mathbb{e}_2$, in which case the corresponding numerator never vanishes. If one would replace D_τ by \mathbb{e}_2 , one would obtain a generalisation of the functions φ_n of [4, Proposition 3.1].

Proof Assume that (2.31) holds. Define $g_{k-2p-2} := 0$. Then, using $[\delta_\tau, D_\tau] = W$ (as follows from Proposition 2.24), it follows that applying δ_τ to a term in the sum in (2.31) yields

$$(-1)^m \frac{D_\tau^m g_{k-2i-2m-2}}{(k-2i-m-1)_m m!} - (-1)^{m-1} \frac{D_\tau^{m-1} g_{k-2i-2m}}{(k-2i-(m-1)-1)_{m-1} (m-1)!}$$

where the second term is taken to be zero when $m = 0$. Also the first term vanishes when $m = p-i$ as g_{k-2p-2} is set to be zero. Hence, after applying δ_τ the sum becomes a telescoping sum, equal to zero. Also $\delta_\tau(g_2 - \mathbb{e}_2 g_0) = g_0 - g_0 = 0$. The third statement follows from the first by the same argument, mutatis mutandis.

Conversely, the first statement follows inductively from the second (or third) by using that all but two terms in the same sum, which is equal to zero, cancel. Hence, these terms are equal.

Inspired by the above result, we now also introduce certain linear combinations which later turn out to be modular. Recall $(x)_n = x(x+1) \cdots (x+n-1)$ denotes the Pochhammer symbol.

Definition 2.32 Let $k \in \mathbb{Z}$, $M \in M_n(\mathbb{Q})$, $\varphi \in \text{Mer}_n^M$ and a family φ as before. For $\ell \in \mathbb{Z}^n$ and $X \in M_{n,2}(\mathbb{Q})$ with $k + |\ell| \geq 0$, the functions $\xi_\ell^X(\varphi)(\tau) : \mathfrak{h} \rightarrow \mathbb{C}$ are defined by

$$\xi_\ell^X(\varphi) := \begin{cases} \sum_{r=0}^{\lfloor \frac{1}{2}(k+|\ell|) \rfloor} (-1)^r \sum_{s=0}^r \frac{D_\tau^r (g_{\ell,s}^{X,r-s})(\varphi)}{(k+|\ell|-r-1)_r (r-s)!} & k + |\ell| \neq 2, \\ g_\ell^X(\varphi) - \mathbb{e}_2 (g_{\ell,0}^{X,1}(\varphi) + g_{\ell,1}^{X,0}(\varphi)) & k + |\ell| = 2. \end{cases} \tag{20}$$

Abbreviate $\xi_\ell^X(\varphi)$ by $\xi_\ell(\varphi)$ if X is the zero matrix.

Remark We formulate all results below for $\xi_\ell^X(\varphi)$ as defined above, but by Proposition 2.31 all results remain valid after replacing $\xi_\ell^X(\varphi)$ by

$$\sum_{r=0}^{\lfloor \frac{1}{2}(k+|\ell|) \rfloor} (-1)^r \sum_{s=0}^r \frac{(D_\tau + \mathfrak{e}_2)^r (g_{\ell,s}^{X,r-s})(\varphi)}{(k + |\ell| - r - \frac{3}{2})_r (r - s)!}.$$

Note that this equation, as well as Eq. 20, can be inverted, expressing $\xi_m^X(\varphi)$ as linear combination of derivatives of certain $g_\ell^X(\varphi)$.

Now, we have completed the full set-up for the main results on Taylor coefficients of quasi-Jacobi forms. In the first theorem, we characterize when a family of meromorphic functions is invariant under the quasimodular action in terms of its Taylor coefficients, generalizing [9, Theorem 3.2]:

Theorem 2.33 *Let Γ be a congruence subgroup, $k \in \mathbb{Z}$ and $M \in M_n(\mathbb{Q})$. As before, let $\varphi = \{\varphi_{i,j}\}$ be a family of meromorphic functions $\mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$ with $\varphi := \varphi_{0,0} \in \text{Mer}_n^M$ and admitting a Laurent expansion around $\mathbf{z} = \mathbf{0}$. Then, the following are equivalent:*

- (i) *The function φ satisfies the quasimodular transformation (8)*

$$(\varphi|_{k,M}\gamma)(\tau, \mathbf{z}) = \sum_{i,j} \varphi_{i,j}(\tau, \mathbf{z}) \left(\frac{c}{c\tau + d}\right)^{i+|j|} \frac{\mathbf{z}^j}{(2\pi i)^i}$$

for all $\gamma \in \Gamma$;

- (ii) *The coefficients $g_\ell(\varphi)$ are quasimodular forms of weight $k + |\ell|$ on Γ , of which the transformation is uniquely determined by the coefficients of the $\varphi_{i,j}$, i.e.*

$$\delta_\tau^r g_\ell(\varphi) = \sum_{s=0}^r \frac{r!}{(r-s)!} g_{\ell,s}^{r-s}(\varphi);$$

- (iii) *The functions $\xi_\ell(\varphi)$ are modular forms of weight $k + |\ell|$ on Γ .*

Proof Expanding (8) yields

$$\sum_{\ell} \frac{g_\ell(\varphi)(\gamma\tau)}{(c\tau + d)^{k+|\ell|}} \mathbf{z}^\ell = \sum_{\ell} \sum_{r,s} \frac{g_{\ell,s}(\varphi)(\tau)}{r!} \left(\frac{c}{c\tau + d}\right)^{r+s} (B_M(\mathbf{z}, \mathbf{z}))^r \mathbf{z}^\ell$$

Extracting on both sides the coefficient of \mathbf{z}^ℓ yields

$$\delta_\tau^r g_\ell(\varphi) = \sum_s \frac{r!}{(r-s)!} g_{\ell,s}^{r-s}(\varphi).$$

Also, the coefficient $g_\ell(\varphi)$ is holomorphic in \mathfrak{h} as well as at the cusps of Γ , because of the analytic properties of the functions $\varphi_{i,j}$. Hence, the first statement is equivalent to the second. The equivalence of the second to the third follows from Proposition 2.31.

Sometimes one is interested in comparing the action of δ_τ on some quasi-Jacobi form φ with the action of δ_τ on the Taylor coefficients of φ . By expanding the equality in (2.33) above we find that the action of δ_τ on the Taylor coefficients of φ corresponds to the action of $B_M(\mathbf{z}, \mathbf{z}) + \delta_\tau + \sum_j z_j \delta_{z_j}$ on φ . In [20, Proposition 2(ii)] an analogous result for the Fourier coefficients of a weakly holomorphic quasi-Jacobi form is derived.

Corollary 2.34 *If $\varphi = \sum_{\ell} g_{\ell}(\varphi)(\tau) \mathbf{z}^{\ell}$ is a strictly meromorphic quasi-Jacobi form, then*

$$\sum_{\ell} \delta_{\tau} g_{\ell}(\varphi) \mathbf{z}^{\ell} = (B_M(\mathbf{z}, \mathbf{z}) + \delta_{\tau} + \sum_j z_j \delta_{z_j}) \varphi.$$

One can characterize a quasi-Jacobi form φ by its Taylor coefficients in three ways: by considering $g_{\ell}^X(\varphi)$ as a vector-valued quasimodular form, by the modularity of the functions $\xi_{\ell}^X(\varphi)$, and, finally, by the action of δ_{τ} on the quasimodular form $g_{\ell}^X(\varphi)$. Write f^X for $g_{\ell}^X(\varphi)$ or $\xi_{\ell}^X(\varphi)$. Then, the ‘elliptic transformation’ of the (quasi)modular form f^X is given by

$$\rho(X') \zeta_{X',X} f^{X+X'} = f^X \quad \text{for all } X' \in M_{n,2}(\mathbb{Z}). \tag{21}$$

Recall that $g_{\ell}^X(\varphi)$ and $\xi_{\ell}^X(\varphi)$ are only defined when the zeros of φ are orthogonal (see Definition 2.28) and depend on a family of functions $\varphi_{i,j}: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$. Moreover, recall that in case φ is a quasi-Jacobi form by Convention 2.26 this family $\varphi = \{\varphi_{i,j}\}$ determines the transformation of φ . Theorem 1.7 follows from the following result.

Theorem 2.35 *Let $k \in \mathbb{Z}, M \in M_n(\mathbb{Q})$ and $\varphi \in \text{Mer}_n^M$ be such that the poles of φ are orthogonal. Given a family $\varphi = \{\varphi_{i,j}\}$, indexed by (i, j) in a finite subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n$, of meromorphic functions $\varphi_{i,j}: \mathfrak{h} \times \mathbb{C}^n \rightarrow \mathbb{C}$ with $\varphi = \varphi_{0,0}$, the following are equivalent:*

- (i) *The function φ is a strictly meromorphic quasi-Jacobi form of weight k and index M for which the functions $\varphi_{i,j}$ determine its transformation behaviour as in (8) and (9).*
- (ii) *For all $X = (\lambda, \mu) \in M_{n,2}(\mathbb{Q})$ with $(\tau, \lambda\tau + \mu)$ not a pole of φ for some $\tau \in \mathfrak{h}$, the function $g_0^X(\varphi)$ is a vector-valued quasimodular form satisfying (21) and transforming as*

$$g_0^X(\varphi)|_{k\gamma} = \sum_s g_{0,s}^{X\gamma}(\varphi) \left(\frac{c}{c\tau + d}\right)^s.$$

- (ii') *For all $X \in M_{n,2}(\mathbb{Q})$ the function $g_{\ell}^X(\varphi)$ is a vector-valued quasimodular form satisfying (21) for $\ell = 0$ and transforming as*

$$g_{\ell}^X(\varphi)|_{k\gamma} = \sum_r \sum_s \frac{1}{(r-s)!} g_{\ell,s}^{X\gamma, r-s}(\varphi) \left(\frac{c}{c\tau + d}\right)^r.$$

- (iii) *For all $\ell \in \mathbb{Z}^n$ the functions $\xi_{\ell}(\varphi)$ are modular forms of weight $k + |\ell|$ for $\text{SL}_2(\mathbb{Z})$ and for all $X \in M_{n,2}(\mathbb{Q})$ the functions $\xi_0^X(\varphi)$ satisfy (21).*
- (iii') *For all $X \in M_{n,2}(\mathbb{Q})$ and $\ell \in \mathbb{Z}^n$ the functions $\xi_{\ell}^X(\varphi)$ in (20) are modular forms of weight $k + |\ell|$ for Γ_X and satisfy (21).*
- (iv) *For all $X \in M_{n,2}(\mathbb{Q})$ and $\ell \in \mathbb{Z}^n$ the functions $g_{\ell}^X(\varphi)$ are quasimodular forms of weight $k + |\ell|$ for Γ_X , satisfying (21) and*

$$\delta_{\tau}^r g_{\ell}^X(\varphi) = \sum_s \frac{r!}{(r-s)!} g_{\ell,s}^{X, r-s}(\varphi). \tag{22}$$

Proof (2.35) implies (2.35): Let $X \in M_{n,2}(\mathbb{Q})$ and $\gamma \in \Gamma_X$ be given. By Proposition 2.27 the function $\varphi||X$ satisfies the conditions of Theorem 2.33 for $\Gamma = \Gamma_X$. Moreover, by the same proposition (21) is satisfied.

(2.35) implies (2.35'): This follows directly from Theorem 2.33 for $\Gamma = \Gamma_X$.

(2.35') implies (2.35): Observe that $\Gamma_X = \text{SL}_2(\mathbb{Z})$ for X equal to the zero matrix. Hence, we simply forget some of the properties of ξ_ℓ^X .

(2.35) implies (2.35'): As ξ_ℓ is a modular form for $\text{SL}_2(\mathbb{Z})$ for all $\ell \in \mathbb{Z}^n$, it follows by Theorem 2.33 that φ satisfies the quasimodular transformation for all $\gamma \in \text{SL}_2(\mathbb{Z})$. As by Proposition 2.27

$$\varphi|X|_k\gamma = \sum_{i,j} (\varphi_{i,j}|X\gamma) \left(\frac{1}{2\pi i} \frac{c}{c\tau + d}\right)^i \left(\frac{c\mathbf{z}}{c\tau + d}\right)^j,$$

the result follows by extracting the coefficients of \mathbf{z} on both sides. Finally, note that $\xi_0^X = g_0^X$.

(2.35') implies (2.35): This follows directly by restricting to $\ell = \mathbf{0}$.

(2.35) implies (2.35): Suppose $\mathbf{z} = \lambda\tau + \mu$, with $X = (\lambda, \mu) \in M_{n,2}(\mathbb{Q})$, is not a pole of φ . Let φ_i be the family of functions corresponding to $\varphi_{i,0}$. Using (2.35), for $\gamma \in \text{SL}_2(\mathbb{Z})$ one has that

$$\begin{aligned} (g_0^X(\varphi)|_k\gamma)(\tau) &= \sum_i g_0^{X\gamma}(\varphi_i)(\tau) \left(\frac{1}{2\pi i} \frac{c}{c\tau + d}\right)^i \\ &= \rho(-X\gamma) \sum_{i,\ell} (\varphi_{i,\ell}|X\gamma)(\tau, 0) \left(\frac{1}{2\pi i} \frac{c}{c\tau + d}\right)^i (\lambda^\gamma)^\ell \\ &= \rho(-X\gamma) \sum_{i,j,\ell} (\varphi_{i,j+\ell}|X\gamma)(\tau, 0) \binom{j+\ell}{j} \left(\frac{c(\lambda^\gamma\tau + \mu^\gamma)}{c\tau + d}\right)^j \lambda^\ell. \end{aligned}$$

On the other hand,

$$\begin{aligned} (g_0^X(\varphi))|_k\gamma &= \rho(-X) \sum_\ell (\varphi_{0,\ell}|X|_k\gamma)(\tau, 0) \lambda^\ell \\ &= \rho(-X\gamma) \sum_\ell (\varphi_{0,\ell}|_k\gamma|X\gamma)(\tau, 0) \lambda^\ell. \end{aligned}$$

Combining the identities yields

$$\sum_\ell (\varphi_{0,\ell}|_k\gamma|X\gamma)(\tau, 0) \lambda^\ell = \sum_{i,j,\ell} (\varphi_{i,j+\ell}|X\gamma)(\tau, 0) \binom{j+\ell}{j} \left(\frac{c(\lambda^\gamma\tau + \mu^\gamma)}{c\tau + d}\right)^j \lambda^\ell.$$

As both sides equal $\rho(X\gamma)g_0^X(\varphi)$, which is periodic with finite period as a function of λ , the constant terms with respect to λ agree. Hence,

$$(\varphi|_\gamma|X)(\tau, 0) = \sum_{i,j} (\varphi_{i,j}|X)(\tau, 0) \left(\frac{1}{2\pi i} \frac{c}{c\tau + d}\right)^i \left(\frac{c(\lambda\tau + \mu)}{c\tau + d}\right)^j$$

for all $X = (\lambda, \mu) \in M_{n,2}(\mathbb{Q})$ with X not corresponding to a pole. Therefore, φ satisfies the quasimodular transformation for all \mathbf{z} of the given form. As $(\mathbb{Q}^n\tau \times \mathbb{Q})^n \setminus P_\varphi$, with P_φ the set of poles of φ , lies dense in \mathbb{C}^n for all $\tau \in \mathfrak{h}$, the function φ satisfies the quasimodular transformation equation.

For the elliptic transformation, we again assume $\mathbf{z} = \lambda\tau + \mu$, with $X = (\lambda, \mu) \in M_{n,2}(\mathbb{Q})$, is not a pole of φ . Given $X' = (\lambda', \mu') \in M_{n,2}(\mathbb{Z})$, by Definitions 2.29, 2.25 and

Proposition 2.3 we have

$$\begin{aligned} \sum_{\ell} (\varphi_{0,\ell}|X)(\tau, 0) \lambda^\ell &= \rho(X) g_0^X(\varphi) \\ &= \rho(X) \rho(X') \zeta_{X',X} g_0^{X+X'}(\varphi) \\ &= \rho(X) \rho(X') \rho(-X - X') \zeta_{X',X} \sum_{\ell} (\varphi_{0,\ell}|(X + X'))(\tau, 0) (\lambda + \lambda')^\ell \\ &= \sum_{\ell} (\varphi_{0,\ell}|X'|X)(\tau, 0) (\lambda + \lambda')^\ell. \end{aligned}$$

The coefficients of λ agree, so

$$(\varphi|X')(\tau, z) = \sum_j \varphi_{0,j}(\tau, z) (-\lambda')^j$$

for all z of the given form. As before by continuity of φ the above equation holds for all z .

Remark The proof of the above result also applies to *weak Jacobi forms*, after replacing ‘strictly meromorphic Jacobi form’ and ‘(quasi)modular’ by ‘weak Jacobi form’ and ‘weakly holomorphic (quasi)modular’, respectively.

Specializing to *holomorphic Jacobi forms* (instead of *meromorphic quasi-Jacobi forms*), we obtain the following result, generalizing the main results on Taylor coefficients of Jacobi forms in [9] to multivariable Jacobi forms.

Corollary 2.36 *Let $k \in \mathbb{Z}$, $M \in M_n(\mathbb{Q})$ and $\varphi \in \text{Hol}_n^M$. Then, the following are equivalent:*

- (i) *The function φ is a holomorphic Jacobi form of weight k and index M .*
- (ii) *For all $X \in M_{n,2}(\mathbb{Q})$ the function $g_0^X(\varphi)$ is a vector-valued modular form satisfying (21) and transforming as*

$$g_0^X(\varphi)|_{k\gamma} = g_0^{X\gamma}(\varphi).$$

- (ii') *For all $X \in M_{n,2}(\mathbb{Q})$ the function $g_\ell^X(\varphi)$ is a vector-valued quasimodular form satisfying (21) for $\ell = \mathbf{0}$ and transforming as*

$$g_\ell^X(\varphi)|_{k\gamma} = \sum_r \frac{1}{r!} g_\ell^{X\gamma,r}(\varphi) \left(\frac{c}{c\tau + d} \right)^r.$$

- (iii') *For all $X \in M_{n,2}(\mathbb{Q})$ and $\ell \in \mathbb{Z}^n$ the functions $\xi_\ell^X(\varphi)$ given by*

$$\xi_\ell^X(\varphi) = \sum_r (-1)^r \frac{D_\tau^r g_\ell^{X,r}(\varphi)}{(k + |\ell| - 1)_r r!}$$

are modular forms of weight $k + |\ell|$ for Γ_X and satisfy (21).

- (iv) *For all $X \in M_{n,2}(\mathbb{Q})$ and $\ell \in \mathbb{Z}^n$ the functions $g_\ell^X(\varphi)$ are quasimodular forms of weight $k + |\ell|$ for Γ_X , satisfying (21) and*

$$\delta_\tau^r g_\ell^X(\varphi) = g_\ell^{X,r}(\varphi).$$

3 Quasimodular algebras for congruence subgroups

The main result of the previous part, Theorem 2.35, will almost immediately imply the proof of Theorem 1.2 on the quasimodularity of the elements of $\Lambda^*(N)$ for some subgroup. We will first introduce a more general set-up, in the context of which we will present this proof. In the rest of this section, we provide many examples of algebras of functions on partitions to which this general set-up applies, i.e. we will recall the hook-length moments and the (double) moment functions and explain how the corresponding algebras can be extended to several congruence subgroups.

3.1 General set-up

We answer the question of how to extend an algebra of functions on partitions for which the q -bracket is a quasimodular form for $SL_2(\mathbb{Z})$ to one for a congruence subgroup Γ . More precisely, we consider *quasimodular algebras* for Γ .

Definition 3.1 A *quasimodular algebra* for a congruence subgroup $\Gamma \leq SL_2(\mathbb{Z})$ is a graded algebra of functions $f: \mathcal{P} \rightarrow \mathbb{C}$ for which $\langle f \rangle_q$ is a quasimodular form for Γ of the same weight as f .

Given a quasimodular algebra for $SL_2(\mathbb{Z})$, we now present a construction of a quasimodular algebra for a congruence subgroup. To do so, from now on we assume that $\Phi: \mathcal{P} \times \mathbb{C}^r \rightarrow \mathbb{C}$ and $k \in \mathbb{Z}$ are such that for all $n \geq 1$ the function $\varphi_n^\Phi: \mathfrak{h} \times M_{n,r}(\mathbb{C}) \rightarrow \mathbb{C}$ given by

$$\varphi_n^\Phi(\tau, Z) := \left\langle \prod_{i=1}^n \Phi(\cdot, Z_i) \right\rangle_q,$$

where Z_i is the i th row of Z , is a meromorphic quasi-Jacobi form of weight kn which admits a Laurent expansion around all $Z \in M_{n,r}(\mathbb{Q})$ (after identifying $M_{n,r}(\mathbb{C})$ with \mathbb{C}^{nr}). Here, $\Phi(\lambda, z)$ can be thought of a generalisation of the generating series $W_\lambda(z)$ of the Bloch–Okounkov functions Q_k , defined by (2).

Definition 3.2 Given such a Φ , for $\mathbf{a} \in \mathbb{Q}^r$ denote by $f_\ell^\Phi(\cdot, \mathbf{a}) = f_\ell(\cdot, \mathbf{a}): \mathcal{P} \rightarrow \mathbb{C}$ the ℓ th Taylor coefficient of $\Phi(z)$ around $z = \mathbf{a}$, i.e.

$$\Phi(\cdot, z) =: \sum_{\ell} f_\ell(\cdot, \mathbf{a}) (z - \mathbf{a})^\ell.$$

Define the graded \mathbb{Q} -algebra $\mathcal{F}^\Phi(N) = \mathcal{F}(N)$ as the algebra generated by the weight $k + |\ell|$ elements $f_\ell(\cdot, \mathbf{a})$ for $\mathbf{a} \in \frac{1}{N}\mathbb{Z}^r$, $\ell \in \mathbb{Z}^r$.

Remark By Theorem 2.35, up to a sign, $f_m(\cdot, \mathbf{a})$ and $f_m(\cdot, \mathbf{b})$ agree whenever $\mathbf{a} - \mathbf{b} \in \mathbb{Z}^r$. Hence, in the definition one can assume that $\mathbf{a} \in [0, 1)^r$.

For example, letting Φ be the Bloch–Okounkov generating series W , we find $Q_{\ell+1}(\mathbf{a}) = e(-\frac{1}{2}\mathbf{a})f_\ell^W(\mathbf{a})$ (see 4) and $\Lambda^*(N) = \mathcal{F}^W(N)$.

Now we relate the Taylor coefficients of Φ to the Taylor coefficients of the corresponding meromorphic quasi-Jacobi forms φ_n^Φ . Let $L \in M_{n,r}(\mathbb{Z})$ and $A \in M_{n,r}(\mathbb{Q})$. An arbitrary monomial f_L in $\mathcal{F}^\Phi(N)$ is given by

$$f_L(A) := f_{L_1}(A_1) \cdots f_{L_n}(A_n) \tag{23}$$

with L_i and A_i the i th row of L and A , respectively. Recall that $g_\ell^X(\varphi)$ denotes the ℓ th Taylor coefficient of φ around $\lambda\tau + \mu$; see Definition 2.29. By construction of these Taylor coefficients, we find

$$\langle f_L^\Phi(A) \rangle_q = g_L^{(0,A)}(\varphi_n^\Phi),$$

where on the right-hand side we identified $M_{n,r}(\mathbb{Z})$ and $M_{n,r}(\mathbb{Q})$ with \mathbb{Z}^{nr} and \mathbb{Q}^{nr} , respectively.⁴

Recall that for $\widehat{N} \in \mathbb{Z}$, we write $m_{\widehat{N}} = \begin{pmatrix} \widehat{N} & 0 \\ 0 & 1 \end{pmatrix}$. The following result is the general statement of Theorem 1.2.

Theorem 3.3 *Given Φ as above and $N \geq 1$, let $\widehat{N} = (2, N)N$. The algebra $\mathcal{F}^\Phi(N)$ is a quasimodular algebra for $m_{\widehat{N}}^{-1}\Gamma(\widehat{N})m_{\widehat{N}}$.*

Proof Consider a monomial element $f_L(A)$ of $\mathcal{F}(N)$ as in (23), for some $L \in M_{n,r}(\mathbb{Z})$ and $A \in M_{n,r}(\mathbb{Q})$. Write $X = (0, A)$. Then, $\langle f_L(A) \rangle_q = g_L^X(\varphi_n)$. This Taylor coefficient is quasimodular for Γ_X by Theorem 2.35. Therefore, it suffices to show that the q -bracket respects the weight grading of $\mathcal{F}(N)$ and that Γ_X contains $m_{\widehat{N}}^{-1}\Gamma(\widehat{N})m_{\widehat{N}}$.

For the first, observe that the weight of f is given by $\sum_{i=1}^n(k + |L_i|)$, whereas correspondingly the weight of $g_L^X(\varphi)$ equals $kn + |L|$ (here $|L| = \sum_{i,j} L_{ij}$).

Write M for the index of φ_n^Φ and $B = B_M$ for the corresponding bilinear form. Recall

$$\Gamma_X = \{\gamma \in \text{SL}_2(\mathbb{Z}) \mid X\gamma - X \in M_{n,2}(\mathbb{Z}), \rho(X - X\gamma) = \zeta_{X, X\gamma - X}\}.$$

Writing $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$X\gamma - X = (cA, (d - 1)A), \quad \rho(X\gamma - X) = e((c^2 - c(d - 1) + (d - 1)^2)B(A, A))$$

and

$$\zeta_{X, X\gamma - X} = e(B(0, (d - 1)A) - B(cA, A)) = e(-cB(A, A)).$$

Observe that $2N^2 B(A, A)$ is integral. Hence, if $\gamma \in \text{SL}_2(\mathbb{Z})$ satisfies

$$c \equiv 0 \pmod{N}, \quad d \equiv 1 \pmod{N} \quad \text{and} \quad c^2 - c(d - 1) + (d - 1)^2 \equiv c \pmod{2N^2}, \quad (24)$$

then $\gamma \in \Gamma_X$.

Let $N' \in \mathbb{Z}_{>0}$. Then,

$$m_{N'}^{-1}\Gamma(N')m_{N'} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N'^2}, a \equiv d \equiv 1 \pmod{N'} \right\}.$$

In case $2 \nmid N$, the conditions (24) are satisfied for all $\gamma \in m_{N'}^{-1}\Gamma(N')m_{N'}$ when $N' = N$, in case $2 \mid N$ for $N' = 2N$. Therefore, $m_{\widehat{N}}^{-1}\Gamma(\widehat{N})m_{\widehat{N}} \leq \Gamma_X$.

⁴Here we hide a small subtlety. Recall that Definition 2.29, in which the Taylor coefficients of a strictly meromorphic Jacobi form are defined, depended not only on this Jacobi form but also on a family of meromorphic functions determined by the transformation of this Jacobi form. We omit this family from the notation, as, for $X = (0, A)$ the double slash operator $\|X$ coincides with the slash operator $|X$, so that the ‘‘Taylor coefficients’’ do not involve this family.

For a monomial element $f_L(A)$ as in (23) with $L \in M_{n,r}(\mathbb{Z})$ and $A \in M_{n,r}(\mathbb{Q})$, and $\gamma \in \Gamma_1(N)$ one has that

$$\langle f_L(A) \rangle_q | \gamma = e((c^2 - cd + (d - 1)^2)B(A, A)) \langle f_L(A) \rangle_q, \tag{25}$$

where B is again the bilinear form corresponding to the index of φ_n^Φ . Hence, restricting to $A \in M_{n,r}(\mathbb{Q}) \simeq \mathbb{Q}^{nr}$ for which $e((c^2 - cd + (d - 1)^2)B(A, A)) = 1$ for all $\gamma \in \Gamma_1(N)$, we find the following result. This result allows us to derive Theorem 1.3, using some additional properties of the Bloch–Okounkov n -point functions.

Proposition 3.4 *Given $N \geq 1$, for all $L \in M_{n,r}(\mathbb{Z})$ and $A \in \frac{1}{N}M_{n,r}(\mathbb{Z})$ satisfying $B(A, A) \in \frac{1}{2}\mathbb{Z} + \frac{1}{N}\mathbb{Z}$, one has that $\langle f_L(A) \rangle_q$ is a quasimodular form for $\Gamma_1(N)$.*

Proof This follows from (25) using the following two observations. First of all, $c^2 - cd + (d - 1)^2 \equiv 0 \pmod N$ when $c \equiv 0, d \equiv 1 \pmod N$. Secondly, for integers c, d the integer $c^2 - cd + (d - 1)^2$ is always even whenever not both c and d are even.

In the Introduction, we introduced certain functions $Q_k^{(p)}$ introduced in [12]. In that work the authors let

$$Q_k^{(p)}(\lambda) = \beta_k^{(p)} + \sum_{\gcd(2\lambda_i - 2i + 1, p) = 1} ((\lambda_i - i + \frac{1}{2})^{k-1} - (-i + \frac{1}{2})^{k-1}),$$

where $\beta_k^{(p)} = \beta_k(0)(1 - \frac{1}{p})$. Observe that for primes p this agrees with the definition in (6) in the Introduction. Similarly, we define functions f_ℓ^d in terms of the Taylor coefficients $f_\ell(\mathbf{a})$.

Definition 3.5 Let Φ be as above. Given $\mathbf{d} \in \mathbb{Z}_{>0}^r$, we let

$$U(\mathbf{d}) = \{0, \frac{1}{d_1}, \dots, \frac{d_1-1}{d_1}\} \times \dots \times \{0, \frac{1}{d_r}, \dots, \frac{d_r-1}{d_r}\}$$

and for $\ell \in \mathbb{Z}^r$ define

$$f_\ell^{d, \Phi}(\lambda) = f_\ell^d(\lambda) := \sum_{\mathbf{a} \in U(\mathbf{d})} f_\ell^\Phi(\lambda, 2\mathbf{a}) \quad (\lambda \in \mathcal{P}).$$

Define the graded algebra $\mathcal{F}^{(N)} = \mathcal{F}^{(N), \Phi}$ as the \mathbb{Q} -algebra generated by the functions $f_\ell^{d, \Phi}$ for all $\ell \in \mathbb{Z}^r$ and $\mathbf{d} \in \mathbb{Z}_{>0}^r$ for which $d_i \mid N$ for all i .

Then, Theorem 1.4 follows directly from the following result.

Theorem 3.6 *Let Φ be as above. Given $N \geq 1$, the algebra $\mathcal{F}^{(N), \Phi}$ is a quasimodular algebra for the congruence subgroup $\Gamma_0(N^2)$.*

Proof Consider the monomial elements $f_L^D := f_{L_1}^{D_1} \dots f_{L_n}^{D_n}$ in $\mathcal{F}^{(N)}$, where $L, D \in M_{nr}(\mathbb{Z})$. Everywhere in this proof we identify $M_{n,r}(\mathbb{Z})$ with \mathbb{Z}^{nr} . Then,

$$f_L^D = \sum_{A \in U(D)} f_L(\cdot, 2A).$$

Now, by part (2.35') in Theorem 2.35, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ one has that

$$\langle f_L(\cdot, 2A) \rangle_q | \gamma = \sum_{r \geq 0} h_r(2cA, 2dA) \left(\frac{c}{c\tau + d} \right)^r,$$

where $h_r(\lambda, \mu) = \sum_{s=0}^r \frac{1}{(r-s)!} g_{\ell, s}^{X, r-s}(\varphi_n)$ and $X = (\lambda, \mu)$. Note that h_r is zero for all but finitely many r .

If $\gamma \in \Gamma_0(N^2)$, then $2cA \in 2NM_{n,r}(\mathbb{Z})$. Hence, for $X = (0, 2dA)$ and $X' = (2bA, 0)$ one has $\rho(X')\zeta_{X',X} = 1$. Therefore,

$$h_r(2cA, 2dA) = h_r(0, 2dA) \quad \text{and} \quad h_r(0, 2dA + B) = h_r(0, 2dA)$$

for all $B \in 2M_{n,r}(\mathbb{Z})$. As $2dA$ ranges over the same values modulo 2 as $2A$ does for $A \in U(D)$, one finds

$$\langle f_L^D \rangle_q | \gamma = \sum_{r \geq 0} \sum_{A \in U(D)} h_r(0, 2A) \left(\frac{c}{c\tau + d} \right)^r$$

for all $\gamma \in \Gamma_0(N^2)$. Hence, $\langle f_L^D \rangle_q$ is a quasimodular form for $\Gamma_0(N^2)$.

Remark In fact, given $\mathbf{d} \in \mathbb{Z}_{>0}^r$ with $d_i \mid N^2$ for all i , and Dirichlet characters χ_1, \dots, χ_r modulo d_1, \dots, d_r , respectively, the series given by

$$\left\langle \sum_{a \in U(\mathbf{d})} \overline{\chi_1}(a_1 d_1) \cdots \overline{\chi_r}(a_r d_r) f_{\ell}^{\Phi}(\cdot, 2\mathbf{a}) \right\rangle_q$$

are quasimodular forms for $\Gamma_0(N^2)$ of character $\chi_1 \cdots \chi_r$.

The rest of this part is devoted to providing examples of quasimodular algebras of higher level, using Theorem 3.6.

3.2 First application: the Bloch–Okounkov theorem of higher level

The results on the Bloch–Okounkov algebra, as stated in the introduction, are proven in this section. The proofs follow almost immediately from the results in the previous section using the properties of the Bloch–Okounkov n -point functions.

Recall the Bloch–Okounkov n -point functions F_n are defined (in Definition 1.1) as follows. For all $n \geq 0$, let \mathfrak{S}_n be the symmetric group on n letters and let

$$F_n(\tau, z_1, \dots, z_n) = \sum_{\sigma \in \mathfrak{S}_n} V_n(\tau, z_{\sigma(1)}, \dots, z_{\sigma(n)}),$$

where the functions V_n are defined recursively by $V_0(\tau) = 1$ and

$$\sum_{m=0}^n \frac{(-1)^{n-m}}{(n-m)!} \theta^{(n-m)}(\tau, z_1 + \dots + z_m) V_m(\tau, z_1, \dots, z_m) = 0.$$

These n -point functions F_n are quasi-Jacobi forms of which we determine the weight and index (or rather the bilinear form uniquely determining a symmetric matrix $M \in M_n(\mathbb{Q})$ which is the index).

Lemma 3.7 *The n -point functions F_n are meromorphic quasi-Jacobi forms of weight n and index $B(\mathbf{z}, \mathbf{z}) = -\frac{1}{2}(z_1 + \dots + z_n)^2$.*

Proof We start with the observation that for all $n \geq 0$ the function $\frac{\Theta^{(n)}(z)}{\Theta(z)}$ is a true meromorphic Jacobi form (of weight n and index $B(z, z) = 0$), in contrast to $\Theta(z)$ itself which is weakly holomorphic. Namely, all poles are given by $z \in \mathbb{Z}\tau + \mathbb{Z}$ and for $z = a\tau + b$ with $a, b \in \mathbb{Q}$, one has that

$$\frac{\Theta^{(n)}(a\tau + b)}{\Theta(a\tau + b)} = \frac{\sum_{v \in \mathbb{F}} v^n e(vb) q^{v^2/2+av}}{\sum_{v \in \mathbb{F}} e(vb) q^{v^2/2+av}} \rightarrow -a,$$

whenever $\text{Im}\tau \rightarrow \infty$, or equivalently $q \rightarrow 0$.

Next, observe that $\Theta(z_1 + \dots + z_n) V_n(z_1, \dots, z_n)$ can be written as a polynomial of weight $n - 1$ in $\frac{\Theta^{(i)}(z_1 + \dots + z_j)}{\Theta(z_1 + \dots + z_j)}$ for $i, j = 1, \dots, n$; a fact which can be proven inductively by its recursion (3). Hence, $\Theta(z_1 + \dots + z_n) V_n(z_1, \dots, z_n)$ is a meromorphic Jacobi form. As $\Theta(z)^{-1}$ is a meromorphic Jacobi form of weight 1 and index given by the bilinear form $B(z, z) = -\frac{1}{2}z^2$, we conclude that F_n is a meromorphic quasi-Jacobi form of weight n and index $B(\mathbf{z}, \mathbf{z}) = -\frac{1}{2}|\mathbf{z}|^2$.

Observe that Theorem 1.2 is a direct corollary of the previous lemma and Theorem 3.3. Also, Theorem 1.4 follows directly from the previous lemma and Theorem 3.6. So, we are only left with the following proof.

Proof of Theorem 1.3 First of all, in case $a \in \mathbb{Z}$, one has $B(\mathbf{a}, \mathbf{a}) = -\frac{1}{2}|\mathbf{a}|^2 \in \frac{1}{2}\mathbb{Z}$. Hence, the result follows directly from Proposition 3.4.

If $a \notin \mathbb{Z}$, for both $\langle Q_k(\mathbf{a}) \rangle_q$ and $\langle Q_1(\mathbf{a}) \rangle_q$ the root of unity in (25) is $e(B(\mathbf{a}, \mathbf{a})) = e(B(a, a))$, so that the subgroup of quasimodularity for their ratio is $\Gamma_1(N)$.

For the holomorphicity in the second case, observe that $\langle Q_1(\mathbf{a}) \rangle_q$ equals $\Theta(a)^{-1}$ up to a constant. Also, $\langle Q_k(\mathbf{a}) \rangle_q$ can be written as a product of Taylor coefficients of the function $\Theta(z_1 + \dots + z_n + a)^{-1}$ and of $\Theta(z_1 + \dots + z_n + a) F_n(z_1 + a_1, \dots, z_n + a_n)$, the latter Taylor coefficients being holomorphic quasimodular forms. Observe that the Taylor coefficients around $z_1 = \dots = z_n = 0$ of

$$\frac{\Theta(a)}{\Theta(z_1 + \dots + z_n + a)}$$

are all polynomials in the holomorphic quasimodular forms $\frac{\Theta^{(i)}(a)}{\Theta(a)}$. Therefore, $\frac{\langle Q_k(\mathbf{a}) \rangle_q}{\langle Q_1(\mathbf{a}) \rangle_q}$ is a holomorphic quasimodular form.

3.3 Second application: hook-length moments of higher level

As a second example, consider the hook-length moments

$$H_k(\lambda) := -\frac{B_k}{2k} + \sum_{\xi \in Y_\lambda} h(\xi)^{k-2} \quad (k \geq 2),$$

where B_k is the k th Bernoulli number, Y_λ denotes the Young diagram of λ , ξ is a cell in this Young diagram, and $h(\xi)$ denotes the hook-length of this cell. By [6, Theorem 13.5] one has that $H_k(\lambda)$ is (up to a constant) equal to the $(k - 2)$ th order Taylor coefficient of $W_\lambda(z) W_\lambda(-z)$. In particular, any homogeneous polynomial in the H_k admits a quasimodular q -bracket. The results in Sect. 3.1 now specialize to the following statements.

(i). For $a \in \mathbb{Q}$ and $k \in \mathbb{Z}_{\geq 2}$, let

$$H_k(\lambda, a) := \tilde{\alpha}_k(a) + \frac{1}{2} \sum_{\xi \in Y_\lambda} (e(a h(\xi)) + (-1)^k e(-a h(\xi))) h(\xi)^{k-2},$$

where

$$\frac{\tilde{\alpha}_{-2}(a)}{z^2} + \frac{\tilde{\alpha}_{-1}(a)}{z} + \sum_{k \geq 0} \tilde{\alpha}_k(a) \frac{z^{k-2}}{(k-2)!} := \frac{1}{8} \sinh\left(\frac{z + 2\pi ia}{2}\right)^{-2}.$$

Denote by $\mathcal{H}(N)$ the algebra generated by the $H_k(\cdot, a)$ with $k \geq 2$ and $a \in \frac{1}{N}\mathbb{Z}$. The algebra $\mathcal{H}(N)$ is graded by assigning to $H_k(\cdot, a)$ weight k . Let $\widehat{N} = (2, N)N$.

Corollary 3.8 *The algebra $\mathcal{H}(N)$ is quasimodular of level \widehat{N} , after scaling τ by \widehat{N} .*

More concretely, for all homogeneous $f \in \mathcal{H}(N)$ of some weight k , the rescaled q -bracket $\langle f \rangle_{q_{\widehat{N}}}$, where $q_{\widehat{N}} = q^{1/\widehat{N}}$, is a quasimodular form of weight k for $\Gamma(\widehat{N})$.

(ii). Given $N \geq 1, \mathbf{k} \in \mathbb{Z}^n$ and $\mathbf{a} \in \frac{1}{N}\mathbb{Z}^n$, write $H_{\mathbf{k}}(\lambda, \mathbf{a}) = H_{k_1}(\lambda, a_1) \cdots H_{k_n}(\lambda, a_n)$.

Corollary 3.9 *For $\mathbf{a} \in \frac{1}{N}\mathbb{Z}^n$ with $|\mathbf{a}| \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}_{\geq 2}^n$ the q -bracket $\langle H_{\mathbf{k}}(\cdot, \mathbf{a}) \rangle_q$ is a quasimodular form of weight $|\mathbf{k}|$ for $\Gamma_1(N)$.*

(iii). Let

$$H_k^t(\lambda) := -\frac{B_k}{2k} t^k + \sum_{\substack{\xi \in Y_\lambda \\ h(\xi) \equiv 0 \pmod t}} h(\xi)^{k-2} \quad (k, t \in \mathbb{Z}_{>0}),$$

which (up to a constant) also occurs in [3]. Denote by $\mathcal{H}^{(N)}$ the algebra generated by the H_k^t for which k is even and $t \mid N$. This algebra is graded by assigning weight k to H_k^t .

Corollary 3.10 *The algebra $\mathcal{H}^{(N)}$ is quasimodular for $\Gamma_0(N^2)$.*

More concretely, for all homogeneous $f \in \mathcal{H}^{(N)}$ of weight k , the q -bracket $\langle f \rangle_q$ is a quasimodular form of weight k for $\Gamma_0(N^2)$.

3.4 Third application: moment functions of higher level

Next, we consider the moment functions in [25]

$$S_k(\lambda) := -\frac{B_k}{2k} + \sum_{i=1}^{\infty} \lambda_i^{k-1} \quad (k \geq 1).$$

The generating series $\mathcal{S}(z): \mathcal{P} \rightarrow \mathbb{Q}$ given by $\mathcal{S}(z) := \frac{1}{2z^2} + \sum_{k \geq 2} S_k \frac{z^{k-2}}{(k-2)!}$ satisfies [22, Corollary 3.3.2]

$$\langle \mathcal{S}(z_1) \cdots \mathcal{S}(z_n) \rangle_q = \frac{1}{2^{n+1}} \sum_{\alpha \in \Pi(n)} \prod_{A \in \alpha} \sum_{s \in \{-1, 1\}^{|A|}} D_\tau^{|A|-1} E_2(\mathbf{s} \cdot \mathbf{z}_A), \tag{26}$$

where $\Pi(n)$ denotes the set of all set partitions of the set $\{1, \dots, n\}$, $|A|$ the cardinality of the set A , and $\mathbf{z}_A = (z_{a_1}, \dots, z_{a_r})$ if $A = \{a_1, \dots, a_r\}$. Hence, the n -point functions (26) are quasi-Jacobi forms of weight $2n$ and index zero. Because the index of the quasi-Jacobi forms (26) is zero, the root of unity in (25) vanishes for all $\gamma \in \text{SL}_2(\mathbb{Z})$. Therefore, the results of Sect. 3.1 specialize to the following results for the groups $\Gamma_1(N)$ and $\Gamma_0(N)$ (rather than $\Gamma(N)$ and $\Gamma_0(N^2)$, respectively).

(i) & (ii). Let

$$S_k(\lambda, a) := \alpha_k(a) + \frac{1}{2} \sum_{i=1}^{\infty} (e(a\lambda_i) + (-1)^k e(-a\lambda_i)) \lambda_i^{k-1} \quad (a \in \mathbb{Q}, k \geq 1),$$

where

$$\frac{\alpha_{-1}(a)}{2\pi iz} + \sum_{k \geq 0} \alpha_k(a) \frac{(2\pi iz)^{k-1}}{(k-1)!} := \frac{1}{2} (e(z+a) - 1)^{-1} \tag{27}$$

(for $k \geq 2$, these values agree with the constants $\tilde{\alpha}_k$ in the previous example). Denote by $\mathcal{S}(N)$ the algebra generated by the $S_k(\cdot, a)$ with $a \in \frac{1}{N}\mathbb{Z}$. Assign to $S_k(\cdot, a)$ weight k .

Corollary 3.11 *The algebra $\mathcal{S}(N)$ is quasimodular for $\Gamma_1(N)$.*

More concretely, for all homogeneous $f \in \mathcal{H}(N)$ of weight k , the q -bracket $\langle f \rangle_q$ is a quasimodular form of weight k for $\Gamma_1(N)$.

(ii). Let

$$S_k^t(\lambda) := -\frac{B_k}{2k}t^k + \sum_{\substack{i \geq 0 \\ \lambda_i \equiv 0 \pmod t}} \lambda_i^{k-1} \quad (k, t \in \mathbb{Z}_{>0}).$$

Denote by $\mathcal{S}^{(N)}$ the algebra generated by the S_k^t for which k is even and $t \mid N$. Assign to S_k^t weight k .

Corollary 3.12 *The algebra $\mathcal{S}^{(N)}$ is quasimodular for $\Gamma_0(N)$.*

More concretely, for all homogeneous $f \in \mathcal{S}^{(N)}$ of weight k , the q -bracket $\langle f \rangle_q$ is a quasimodular form of weight k for $\Gamma_0(N)$.

3.5 Fourth application: double moment functions of higher level

As a final example, consider the double moment functions introduced in [22] given by

$$T_{k,l}(\lambda) := C_{k,l} + \sum_{m=1}^{\infty} m^k \mathcal{F}_l(r_m(\lambda)) \quad (k \geq 0, l \geq 1).$$

Here, $C_{k,l}$ is a constant equal to $-\frac{B_{k+l}}{2(k+l)}$ if $k = 0$ or $l = 1$ and 0 else, \mathcal{F}_l is the *Seki–Bernoulli polynomial* of positive integer degree l , defined by $\mathcal{F}_l(n) := \sum_{i=1}^n i^{l-1}$ for all $n \in \mathbb{Z}_{>0}$, and the *multiplicity* $r_m(\lambda)$ of parts of size m in a partition λ is defined as the number of parts of λ of size m . The generating series $\mathcal{T}(z, w) := -\frac{1}{2z} - \frac{1}{2w} + \sum_{k+l \equiv 0(2)} T_{k,l} \frac{z^k w^{\ell-1}}{(k)!(\ell-1)!}$ satisfies [22, Theorem 4.4.1.]

$$G_1(z, w) := \langle \mathcal{T}(z, w) \rangle_q = -\frac{1}{2} \frac{\Theta(z+w)}{\Theta(z)\Theta(w)}.$$

Hence, the 1-point function G_1 is a Jacobi form of weight 1 and index $B((z, w), (z, w)) = zw$. This example now deviates from the previous ones because $\langle \mathcal{T}(z_1, w_1) \cdots \mathcal{T}(z_n, w_n) \rangle_q$ is not a Jacobi form of some *fixed* weight (but rather a linear combination of functions of different weights). It is, therefore, that we have to consider a different product \odot on the space of functions of partitions, for which

$$G_n(\mathbf{z}, \mathbf{w}) := \langle \mathcal{T}(z_1, w_1) \odot \cdots \odot \mathcal{T}(z_n, w_n) \rangle_q = (-1)^n \prod_{i=1}^n \frac{1}{2} \frac{\Theta(z_i + w_i)}{\Theta(z_i)\Theta(w_i)}. \tag{28}$$

To define this product consider the isomorphism $\mathbb{C}^{\mathcal{P}} \rightarrow \mathbb{C}[[u_1, u_2, \dots]], f \mapsto \langle f \rangle_{\mathbf{u}}$, given by

$$\langle f \rangle_{\mathbf{u}} := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) u_{\lambda}}{\sum_{\lambda \in \mathcal{P}} u_{\lambda}} \quad (u_{\lambda} = u_{\lambda_1} u_{\lambda_2} \cdots).$$

Then, the *induced product* is defined by

$$\langle f \odot g \rangle_u := \langle f \rangle_u \langle g \rangle_u. \tag{29}$$

The algebra \mathcal{T} generated by the $T_{k,l}$ contains exactly the same elements as the algebra consisting of polynomials in the moment functions $T_{k,l}$ with multiplication given by the induced product \odot [22].

In Sect. 3.1, one can replace the pointwise product on functions of partitions by the product \odot . Therefore, we have the following generalisations of the algebra \mathcal{T} .

(i) For $a, b \in \mathbb{Q}$, and $k \geq 0, \ell \geq 1$, let

$$T_{k,\ell}(\lambda, a, b) := C_{k,\ell}(a, b) + \sum_{m=1}^{\infty} m^k (\mathbf{e}(am) \mathcal{F}_\ell^b(r_m(\lambda)) + (-1)^{k+\ell} \mathbf{e}(-am) \mathcal{F}_\ell^{-b}(r_m(\lambda))),$$

where

$$C_{k,\ell}(a, b) := \begin{cases} \alpha_k(a) & \ell = 1, \\ \alpha_{\ell-1}(b) & k = 0, \\ 0 & \text{else,} \end{cases}$$

and the constants α_k are defined by (27). Also, \mathcal{F}_ℓ^b is defined by $\mathcal{F}_\ell^b(n) := \sum_{i=1}^n \mathbf{e}(bi) i^{\ell-1}$ for all $n \in \mathbb{Z}_{>0}$. Let the \mathbb{Q} -algebra $\mathcal{T}(N)$ be generated by the functions $T_{k,\ell}(\cdot, a, b)$, where $a, b \in \frac{1}{N}\mathbb{Z}$, under the induced product. Assign to $T_{k,\ell}(\cdot, a, b)$ weight $k + \ell$ and extend to a weight grading under the induced product. Let $\widehat{N} = (2, N)N$.

Corollary 3.13 *The algebra $\mathcal{T}(N)$ is quasimodular of level \widehat{N} , after scaling τ by \widehat{N} .*

More concretely, for all homogeneous $f \in \mathcal{T}(N)$ of some weight k , the rescaled q -bracket $\langle f \rangle_{q_{\widehat{N}}}$, where $q_{\widehat{N}} = q^{1/\widehat{N}}$, is a quasimodular form of weight k for $\Gamma(\widehat{N})$.

(ii)

Corollary 3.14 *Let $\mathbf{k}, \ell \in \mathbb{Z}^n$ and $\mathbf{a}, \mathbf{b} \in \frac{1}{N}\mathbb{Z}^n$. Whenever $\mathbf{a} \cdot \mathbf{b} \in \frac{1}{2}\mathbb{Z} + \frac{1}{N}\mathbb{Z}$, one has that*

$$\langle T_{k_1,\ell_1}(\cdot, a_1, b_1) \odot \cdots \odot T_{k_n,\ell_n}(\cdot, a_n, b_n) \rangle_q$$

is a quasimodular form of weight $|\mathbf{k}| + |\ell|$ for $\Gamma_1(N)$.

(iii) Let

$$T_{k,\ell}^{s,t}(\lambda) := C_{k,\ell} + \sum_{m=1}^{\infty} m^k \mathcal{F}_\ell \left(\left\lfloor \frac{r_{ms}(\lambda)}{t} \right\rfloor \right) \quad (k, \ell, s, t \in \mathbb{Z}_{>0}).$$

Denote by $\mathcal{T}^{(N)}$ the algebra generated by the $T_{k,\ell}^{s,t}$, where k, ℓ are even and $s, t \mid N$, under the induced product. Assign to $T_{k,\ell}^{s,t}$ weight $k + \ell$ and extend to a weight grading under the induced product.

Corollary 3.15 *The algebra $\mathcal{T}^{(N)}$ is quasimodular for $\Gamma_0(N^2)$.*

More concretely, for all homogeneous $f \in \mathcal{H}^{(N)}$ of weight k , the q -bracket $\langle f \rangle_q$ is a quasimodular form of weight k for $\Gamma_0(N^2)$.

Remark Combining this result with Corollary 3.14, by restricting to $t = 1$, we find that any polynomial in $T_{k,\ell}^{s,1}$ with respect to the induced product is quasimodular for $\Gamma_0(N)$. A similar statement holds after restricting to $s = 1$.

4 When is the q -bracket modular?

We first state and prove our answer to the question in the title of this section in full generality, using the main result on the Taylor coefficients of quasi-Jacobi forms (Theorem 2.35). Afterwards, we provide many examples, i.e. we prove the results on the ‘modular subspace’ of the Bloch–Okounkov algebra as stated in the introduction, and state similar results for the Bloch–Okounkov algebra for congruence subgroups as well as the algebra of double moment functions.

4.1 Construction of functions with modular q -bracket

Recall $D_\tau = \frac{1}{2\pi i} \frac{\partial}{\partial \tau}$, $e_2 = \frac{1}{12} - 2 \sum_{m,r \geq 1} m q^{mr}$ is the quasimodular Eisenstein series of weight 2 and $g_{\ell,s}^{r-s}(\varphi)$ is a ‘Taylor coefficient’ of φ defined by Definition 2.29. Given a quasi-Jacobi form φ of weight k satisfying the conditions of Theorem 2.35, the functions

$$\xi_\ell(\varphi) = \sum_r (-1)^r \sum_{s \leq r} \frac{(D_\tau + e_2)^r g_{\ell,s}^{r-s}(\varphi)}{(k + |\ell| - r - \frac{3}{2})_r (r - s)!}$$

are modular forms (see also Proposition 2.31). Therefore, as in the previous section, assume that $\Phi: \mathcal{P} \times \mathbb{C}^r \rightarrow \mathbb{C}$ and $k \in \mathbb{Z}$ are such that for all $n \geq 1$ the function $\varphi_n^\Phi: \mathfrak{h} \times M_{n,r}(\mathbb{C}) \rightarrow \mathbb{C}$ given by

$$\varphi_n^\Phi(\tau, Z) := \left\langle \prod_{i=1}^n \Phi(\cdot, Z_i) \right\rangle_q,$$

where Z_i is the i th row of Z , is a meromorphic quasi-Jacobi form of weight kn which admits a Laurent expansion around all $Z \in M_{n,r}(\mathbb{Q})$ (after identifying $M_{n,r}(\mathbb{C})$ with \mathbb{C}^{nr}). Write $\mathcal{F} = \mathcal{F}^\Phi(1)$ for the graded algebra of Taylor coefficients of Φ (see Definition 3.2). Given $\ell \in \mathbb{Z}^n$, our aim is to find $h_\ell \in \mathcal{F}$ such that $\langle h_\ell \rangle_q = \xi_\ell(\varphi_n)$, i.e. we want to determine a pullback of $\xi_\ell(\varphi_n)$ under the q -bracket. Observe that these pullbacks h_ℓ for all ℓ define a unique linear map $\mathcal{F} \rightarrow \mathcal{F}$ so that $\pi(f_\ell) = h_\ell$. Note that in this case $\langle \pi(f) \rangle_q$ is modular for all $f \in \mathcal{F}$.

To define π , assume the algebra \mathcal{F} satisfies the following two properties:

- (i) $Q_2 \in \mathcal{F}$,
- (ii) There is a linear operator \mathcal{D} acting on \mathcal{F} such that

$$\left\langle \mathcal{D} \prod_{i=1}^n \Phi(\mathbf{z}^i) \right\rangle_q = (\delta_\tau + z_1 \delta_{z_1} + \dots + z_n \delta_{z_n}) \varphi, \tag{30}$$

where \mathcal{D} is extended to a linear operator on $\mathcal{F}[[z_1, \dots, z_n]]$ by $\mathcal{D}(f \mathbf{z}^\ell) = \mathcal{D}(f) \mathbf{z}^\ell$ for all $f \in \mathcal{F}$ and $\ell \in \mathbb{Z}^n$.

Remark As observed in [25], by an easy computation one finds that the function Q_2 makes the q -bracket equivariant with respect to the operator $D_\tau + e_2$, i.e.

$$\langle Q_2 f \rangle_q = (D + e_2) \langle f \rangle_q$$

for all $f: \mathcal{P} \rightarrow \mathbb{C}$. This motivates the first condition. The second condition is motivated by noting that

$$g_{\ell,s}(\varphi) = g_{\ell} \left(\sum_{i+|j|=s} \varphi_{i,j} \frac{z^j}{(2\pi i)^i} \right) = \frac{1}{s!} g_{\ell} \left((\delta_{\tau} + z_1 \delta_{z_1} + \dots + z_n \delta_{z_n})^s \varphi \right). \tag{31}$$

Recall that the algebra $\mathcal{F} = \mathcal{F}^{\Phi}(1)$ is generated by the Taylor coefficients $f_{\ell} = f_{\ell}(\mathbf{0})$ of Φ for $\ell \in \mathbb{Z}^r$. An arbitrary monomial f_L in \mathcal{F} is given by $f_{L_1} \cdots f_{L_n}$ with $L \in M_{n,r}(\mathbb{Z})$. We define an operator on \mathcal{F} corresponding to the index of φ .

Definition 4.1 Let $M = (m_{i,j}) \in M_n(\mathbb{Q})$ be the index of φ_n^{Φ} . Define \mathcal{M} to be the linear operator on \mathcal{F} which is given on monomials by

$$\mathcal{M}f_L := \sum_{i,j} m_{i,j} f_{L - e_i - e_j} \quad (L \in M_{n,r}(\mathbb{Z}) \simeq \mathbb{Z}^{nr}),$$

where, on the right-hand side, e_i is a unit vector in \mathbb{Z}^{nr} .

Now, Theorem 1.5 can more explicitly be stated as follows, where the three properties below should be compared with the three properties satisfied by the functions h_k in the introduction (Sect. 1.2).

By M denote the algebra of modular forms for $SL_2(\mathbb{Z})$ with rational Fourier coefficients.

Theorem 4.2 Let \mathcal{F} be a quasimodular algebra satisfying the above conditions, and $\mathcal{M}, \mathcal{D} : \mathcal{F} \rightarrow \mathcal{F}$ the operators defined by Definition 4.1 and Eq. 30, respectively. Then, the linear mapping $\pi : \mathcal{F} \rightarrow \mathcal{F}$ given by

$$\pi(f) = \sum_{r \geq 0} \sum_{s=0}^r (-1)^r \frac{Q_2^r \mathcal{M}^{r-s} \mathcal{D}^s f}{(m - r - \frac{3}{2})_r (r - s)! s!}$$

on $f \in \mathcal{F}$ of homogeneous weight m satisfies

$$\langle \pi(\mathcal{F}) \rangle_q \subseteq M.$$

Furthermore, one can choose a vector subspace $\mathcal{M} \subseteq \pi \mathcal{F}$ such that

- (i) $\mathcal{F} = \mathcal{M} \oplus Q_2 \mathcal{F}$;
- (ii) $\langle \mathcal{M} \rangle_q \subseteq M$;
- (iii) $\langle Q_2 \mathcal{F} \rangle_q \cap M = \{0\}$.

Remark For the Bloch–Okounkov algebra Λ^* the mapping π turns out to be the canonical projection of Λ^* on \mathcal{H} in [21] and $\mathcal{M} = \mathcal{H}$. We do not expect that the conditions in this section ensure that π is a projection in general, nor that $\pi(Q_2 \mathcal{F}) = \{0\}$ (in which case one could choose $\mathcal{M} = \pi(\mathcal{F})$), nor that the splitting in (4.2) is canonical. However, once one has chosen \mathcal{M} it follows immediately from (4.2) that every element of \mathcal{F} has a canonical expansion

$$f = \sum_{i \geq 0} f_i Q_2^i$$

with $f_i \in \mathcal{M}$, and that $\langle f \rangle_q \in M$ precisely if $\langle f_i \rangle_q = 0$ for all $i > 0$.

Proof By (31) and by construction of \mathcal{M} , one has that

$$\frac{1}{s!} \langle \mathcal{M}^{r-s} \mathcal{D}^s f \rangle_q = g_{\ell,s}^{r-s}(\varphi).$$

Hence,

$$\langle \pi(f) \rangle_q = \xi_{\ell}(\varphi),$$

where φ is the meromorphic Jacobi form $\varphi(\mathbf{z}) = \varphi_n^{\Phi}(\mathbf{z}) = \langle \prod_{i=1}^n \Phi(\mathbf{z}^i) \rangle_q$. Hence, $\pi(f)$ is modular under the q -bracket for all $f \in \mathcal{F}$.

Choose $\mathcal{M}^0 \subseteq \mathcal{F}$ such that $\mathcal{F} = \mathcal{M}^0 \oplus Q_2 \mathcal{F}$. Then, we take $\mathcal{M} = \pi \mathcal{M}^0$. As, by definition, $\pi(f) - f$ is a multiple of Q_2 , the first property follows. The second property is immediate as $\mathcal{M} \subseteq \pi(\mathcal{F})$. For the last property, let $f \in Q_2 \mathcal{F}$ with $\langle f \rangle_q \in M$ be given. As f is divisible by Q_2 , the q -bracket $\langle f \rangle_q$ is in the image of $D + \mathfrak{e}_2$ acting on quasimodular forms. Now, the zero function is the only function in the image of $D + \mathfrak{e}_2$ which is modular, so that the last property follows.

We now provide several examples of quasimodular algebras to which Theorem 4.2 applies.

4.2 First example: the Bloch–Okounkov algebra

To apply the results of the previous section to the Bloch–Okounkov algebra Λ^* , we have to understand how the operators δ_{τ} and δ_{z_i} act on the n -points functions F_n (defined by Definition 1.1), or equivalently, we have to understand the transformation behaviour of F_n . This behaviour is uniquely determined by the following two properties.

Proposition 4.3 *For all $n \geq 1$ one has*

$$\begin{aligned} \delta_{\tau} F_n(z_1, \dots, z_n) &= 0, \\ \delta_{z_1} F_n(z_1, \dots, z_n) &= \sum_{i=2}^n F_{n-1}(z_1 + z_i, z_2, z_3, \dots, z_{i-1}, z_{i+1}, z_{i+2}, \dots, z_n). \end{aligned}$$

Remark As F_n is symmetric in its arguments, above proposition provides an expression for $\delta_{z_j} F_n(z_1, \dots, z_n)$ for all j .

Proof The second equality is equivalent to [1, Theorem 0.6], whereas the first statement seems not to be in the literature. As both statements follow by more or less the same argument, we give both proofs. That is, we prove

$$\begin{aligned} \delta_{\tau} V_n(z_1, \dots, z_n) &= 0, \\ \delta_{z_i} V_n(z_1, \dots, z_n) &= \begin{cases} V_{n-1}(z_1, \dots, z_i + z_{i+1}, \dots, z_n) & i < n \\ 0 & i = n \end{cases} \end{aligned} \tag{32}$$

inductively using the recursion (3), from which the proposition follows directly.

For $n = 1$ both statements are clearly true. Hence, by the identity

$$[\delta_{\tau}, D_z^m] = -2mD_z^{m-1}\delta_z + m(m-1)D_z^{m-2}$$

and after assuming that $\delta_{\tau} V_n = 0$, we find that $\theta(z_1 + \dots + z_{n+1}) \delta_{\tau} V_{n+1}(\mathbf{z})$ equals

$$-\sum_{m=0}^{n-1} \frac{(-1)^{n-1-m}}{(n-1-m)!} \theta^{(n-1-m)}(z_1 + \dots + z_m) V_m(z_1, \dots, z_m).$$

By the recursion (3) one finds that this expression vanishes. Hence, $\delta_\tau V_{n+1} = 0$ and $\delta_\tau F_n = 0$ as desired.

Denote

$$V_{m-1}^i(z_1, \dots, z_m) = V_{m-1}(z_1, \dots, z_i + z_{i+1}, \dots, z_m).$$

By applying δ_{z_i} to the recursion (3), using the identity $[\delta_z, D_z^m] = -2mD_z^{m-1}I$ (with I the index operator; see Sect. 2.5) and assuming that (32) holds, we find

$$\begin{aligned} & \theta(z_1 + \dots + z_{n+1}) \delta_{z_i} V_{n+1}(z_1, \dots, z_{n+1}) \\ &= \sum_{m=i}^n \frac{(-1)^{n-m}}{(n-m)!} \theta^{(n-m)}(z_1 + \dots + z_m) V_m(z_1, \dots, z_m) + \\ & \quad - \sum_{m=i+1}^n \frac{(-1)^{n+1-m}}{(n+1-m)!} \theta^{(n+1-m)}(z_1 + \dots + z_m) V_{m-1}^i(z_1, \dots, z_m) \\ &= - \sum_{m=0}^{i-1} \frac{(-1)^{n-m}}{(n-m)!} \theta^{(n-m)}(z_1 + \dots + z_m) V_m(z_1, \dots, z_m) + \\ & \quad - \sum_{m=i+1}^n \frac{(-1)^{n+1-m}}{(n+1-m)!} \theta^{(n+1-m)}(z_1 + \dots + z_m) V_{m-1}^i(z_1, \dots, z_m) \\ &= \theta(z_1 + \dots + z_{n+1}) V_n(z_1, \dots, z_i + z_{i+1}, \dots, z_{n+1}). \end{aligned}$$

Next, recall the j th order differential operators \mathcal{D}_j in [21].

Definition 4.4 Define the j th order differential operators \mathcal{D}_j by

$$\mathcal{D}_j := \sum_{i \in \mathbb{Z}_{\geq 0}^j} \binom{|i|}{i_1, i_2, \dots, i_j} Q_{|i|} \partial_i, \quad \text{with} \quad \partial_i := \frac{\partial^j}{\partial Q_{i_1+1} \partial Q_{i_2+1} \dots \partial Q_{i_j+1}},$$

where the coefficient is a multinomial coefficient (in this section we pretend these operators act on Λ^* , although formally there are only defined on the formal algebra $\mathcal{R} = \mathbb{Q}[Q_1, Q_2, \dots]$ freely generated by the variables Q_1, Q_2, \dots , which admits a canonical mapping to Λ^*).

These operators turn out to correspond to certain symmetric powers of the derivative operators δ_{z_i} . In particular, observe that the coefficient of \mathbf{z}^ℓ in the case $j = 1$ below, is given by $g_{\ell,1}(F_n)$. Hence, the operator \mathcal{D} , requested in the previous section, is given by $\mathcal{D} = \mathcal{D}_2/2$.

Proposition 4.5 For all $j \geq 1$ one has

$$\langle \mathcal{D}_j W(\mathbf{z}) \rangle_q = j! \left(z_1 \delta_{z_1}^{j-1} + \dots + z_n \delta_{z_n}^{j-1} \right) F_n(\mathbf{z}).$$

Proof Observe that

$$\mathcal{D}_j Q_\ell = j! \sum_{i_1 < \dots < i_j} \binom{\ell_{i_1} + \dots + \ell_{i_j} - j}{\ell_{i_1} - 1, \dots, \ell_{i_j} - 1} Q_{\ell_{i_1} + \dots + \ell_{i_j} - j} \widehat{Q}_{i_1} \dots \widehat{Q}_{i_j} \dots,$$

where the binomial coefficient vanishes whenever $l_i = 0$ for some i , and we wrote a hat above an element to indicate that this element is omitted. Hence,

$$\mathcal{D}_j W_\lambda(\mathbf{z}) = j! \sum_{i_1 < \dots < i_j} (z_{i_1} + \dots + z_{i_j}) W_\lambda(z_{i_1} + \dots + z_{i_j}, \dots, \hat{z}_{i_1}, \dots, \hat{z}_{i_j}, \dots),$$

where $\dots, \hat{z}_{i_1}, \dots, \hat{z}_{i_j}, \dots$ stands for the elements z_1, \dots, z_n omitting z_{i_1}, \dots, z_{i_j} . By symmetry of F_n and Proposition 4.3 the statement follows.

Now, by invoking Theorem 4.2, the following refinement of Theorem 1.5 follows. In particular, we find a different expression for the basis elements h_λ in [21], which equal $\pi(Q_{\lambda_1} \dots Q_{\lambda_n})$. Moreover, by construction, we deduce that the q -bracket of h_λ is given by $\xi_{\lambda^+}(F_n)$, where $\lambda^+ = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1)$.

Define the mapping \vee as the algebra homomorphism uniquely determined by $Q_n \mapsto \Delta_n$, where the commuting family of operators Δ_k of [21] are given by

$$\Delta_n := \sum_{i=0}^n (-1)^i \binom{n}{i} \partial^i \mathcal{D}_{n-i}.$$

We write $(x)_n^- = x(x-1) \dots (x-n+1)$ for the falling factorial.

Theorem 4.6 *Let $\partial = \mathcal{D}_1$ and \mathcal{D}_2 be given by Definition 4.4. The linear mapping $\pi : \Lambda^* \rightarrow \Lambda^*$ given by*

$$\pi(f) = \sum_{r \geq 0} \sum_{s=0}^r (-1)^s \frac{Q_2^r \partial^{2r-2s} \mathcal{D}_2^s(f)}{2^r (\ell - r - \frac{3}{2})_r (r-s)! s!}$$

whenever f is of weight ℓ is a projection satisfying $\pi(Q_2 \Lambda^*) = 0$ and

$$\pi(f) = \frac{Q_2^{-3/2+\ell} f^\vee Q_2^{3/2}}{\left(\frac{3}{2}\right)_\ell^- \ell!}. \tag{33}$$

Furthermore, $\mathcal{M} := \pi(\Lambda^*)$ satisfies

- (i) $\Lambda^* = \mathcal{M} \oplus (Q_2)$, where $(Q_2) = Q_2 \Lambda^*$;
- (ii) $\langle \mathcal{M} \rangle_q \subseteq M$;
- (iii) $\langle (Q_2) \rangle_q \cap M = \{0\}$;
- (iv) $\mathcal{M} = \mathcal{H}$, where \mathcal{H} is the harmonic subspace of Λ^* as in [21].

Proof Equation 33, as well as the fact that π is a projection, follows directly from [21, Corollary 2]. In particular, as $\Delta_2(Q_2^{3/2}) = 0$, it follows that $\pi(Q_2 f) = 0$.

For the properties of \mathcal{M} , observe that \mathcal{D} and \mathcal{M} , defined in the previous section, can also be expressed as $\mathcal{D}_2/2$ and $-\partial^2/2$, respectively. Hence, the properties follow from Theorem 4.2, and the fourth follows as the set $\{\pi(Q_\lambda)\}$, where λ goes over all partitions with all parts at least 3, is a basis for both spaces.

4.3 Second example: the Bloch–Okounkov algebra of higher level

To extend the result in the previous section to the Bloch–Okounkov algebras $\Lambda^*(N)$ of level N (see (5) and Sect. 3.2), we should generalize the operators ∂ and \mathcal{D}_2 .

Definition 4.7 Let $\widehat{\mathcal{R}} = \mathbb{Q}[Q_k(a) \mid k \in \mathbb{Z}_{\geq 0}, a \in \mathbb{Q}]$ be the algebra in the formal variables $Q_k(a)$ with canonical projection to $\bigcup_{N \in \mathbb{Z}} \Lambda^*(N)$. Given $\mathbf{a} \in \mathbb{Q}^j$, define the j th order

differential operators \mathcal{D}_j on $\hat{\mathcal{R}}$ by

$$\mathcal{D}_j := \sum_{\mathbf{i} \in \mathbb{Z}_{\geq 0}^j} \sum_{\mathbf{a} \in \mathbb{Q}^j} \binom{|\mathbf{i}|}{i_1, i_2, \dots, i_j} Q_{|\mathbf{i}|}(|\mathbf{a}|) \partial_{\mathbf{i}}(\mathbf{a}),$$

where

$$\partial_{\mathbf{i}}(\mathbf{a}) := \frac{\partial^j}{\partial Q_{i_1+1}(a_1) \cdots \partial Q_{i_j+1}(a_j)}.$$

From now on, we pretend that these operators act on $\Lambda^*(N)$, by identifying $\Lambda^*(N)$ with a quotient of $\hat{\mathcal{R}}$ via the obvious inclusion map. Note that restricted to Λ^* the operators \mathcal{D}_j are the same as defined in Definition 4.4. Similarly, the operators \mathcal{D}_j satisfy the following property.

Proposition 4.8 *For all $j \geq 1$ and $\mathbf{a} \in \mathbb{Q}^n$ one has*

$$\langle \mathcal{D}_j W(\mathbf{z} + \mathbf{a}) \rangle_q = j! \left((z_1 + a_1) \delta_{z_1}^{j-1} + \dots + (z_n + a_n) \delta_{z_n}^{j-1} \right) F_n(\mathbf{z} + \mathbf{a}).$$

Denote by $M(N)$ the algebra of modular forms of level N . Then, Theorem 4.2 specializes to the following result.

Theorem 4.9 *Let $N \geq 1$ and $\partial = \mathcal{D}_1$ and $\mathcal{D}_2: \Lambda^*(N) \rightarrow \Lambda^*(N)$ be given by Equation 4.7. Let the projection $\pi: \Lambda^*(N) \rightarrow \Lambda^*(N)$ be given by*

$$\pi(f) = \sum_{r \geq 0} \sum_{s=0}^r (-1)^s \frac{Q_2^r \partial^{2r-2s} \mathcal{D}_2^s(f)}{2^r (\ell - r - \frac{3}{2})_r (r-s)! s!},$$

whenever $f \in \Lambda^(N)$ is homogeneous of weight ℓ . Then, the subspace $\mathcal{M}(N) := \pi(\Lambda^*(N))$ of $\Lambda^*(N)$ satisfies*

- (i) $\Lambda^*(N) = \mathcal{M}(N) \oplus Q_2 \Lambda^*(N)$;
- (ii) $\langle \mathcal{M}(N) \rangle_q \subseteq M(N)$;
- (iii) $\langle Q_2 \Lambda^*(N) \rangle_q \cap M(N) = \{0\}$.

4.4 Third example: double moment functions

For the algebra of double moments functions (see Sect. 3.5) the n -point functions with respect to the induced product \odot (see 29) are given by (28), i.e.

$$G_n(\mathbf{z}, \mathbf{w}) = \prod_{i=1}^n \frac{\Theta(z_i + w_i)}{\Theta(z_i)\Theta(w_i)},$$

which is a Jacobi form. Hence, the operators $\delta_\tau, \delta_{z_i}$ and δ_{w_i} vanish acting on G_n , so that \mathcal{D} can be taken to be the zero operator. We write δ for the derivation on \mathcal{T} with respect to the induced product (i.e. $\delta(f \odot g) = \delta(f) \odot g + f \odot \delta(g)$ for all $f, g \in \mathcal{T}$) given by

$$\delta(T_{k,\ell}) := \begin{cases} k(\ell - 1)T_{k-1,\ell-1} & k \geq 1, \ell \geq 2, \\ -\frac{1}{2} & k + \ell = 2, \\ 0 & \text{else.} \end{cases}$$

The notation δ is suggested by the fact that $\langle \delta f \rangle_q = \delta_\tau \langle f \rangle_q$ for all $f \in \mathcal{T}$.

Theorem 4.10 *Let the projection $\pi : \mathcal{T} \rightarrow \mathcal{T}$ be given by*

$$\pi(f) = \sum_{r \geq 0} \frac{2^r}{r!} \overbrace{T_{1,1} \odot \cdots \odot T_{1,1}}^r \odot \delta^r(f).$$

Then, $\mathcal{M} = \pi(\mathcal{T})$ satisfies the following three properties:

- (i) $\mathcal{T} = \mathcal{M} \oplus (T_{1,1})$, where $(T_{1,1}) = T_{1,1} \odot \mathcal{T}$;
- (ii) $\langle \mathcal{M} \rangle_q = M$;
- (iii) $\langle T_{1,1} \odot \mathcal{T} \rangle_q \cap M = \{0\}$.

Proof The statement follows along the same lines as Theorem 4.2, by making the following observations:

- Analogous to $\xi_\ell(\varphi)$, the functions

$$\sum_{r \leq |\ell|} (-1)^r \sum_{s \leq r} \frac{\mathfrak{E}_2^r \mathfrak{E}_{\ell,s}^{r-s}(\varphi)}{(r-s)!}$$

are modular forms exactly if φ is a quasi-Jacobi form;

- $\langle T_{1,1} \odot f \rangle_q = -2\mathfrak{E}_2 \langle f \rangle_q$ for all $f \in \mathbb{C}^{\mathcal{P}}$;
- The operator δ coincides with the operator \mathcal{M} ;
- By [22, Theorem 3.4.1], we have that $\langle \mathcal{T} \rangle_q = \widetilde{M}$, from which it follows that equality holds in (4.10).

Remark In fact, for all $f, g \in \mathcal{T}$ one has

$$\pi(f \odot g) = \pi(f) \odot \pi(g).$$

Hence, $\langle \pi \mathcal{T} \rangle_q$ is uniquely determined by $\langle \pi(T_{2,0}) \rangle_q = \langle \pi(T_{1,1}) \rangle_q = 0$ and

$$\langle \pi(T_{k,l}) \rangle_q = \begin{cases} \vartheta^{k-1} G_{l-k+2} & l \geq k \\ \vartheta^l G_{k-l} & k \geq l + 2 \end{cases}$$

for $T_{k,l} \in \mathcal{T}$ with $k + l \geq 4$, where $\vartheta := D_\tau - \mathfrak{E}_2 W$ denotes the Serre derivative on the space of (quasi)modular forms (recall W is the operator multiplying a quasimodular form by its weight) and $G_k = \frac{(k-1)!}{2(2\pi i)^k} e_k$. For any modular form f , the Serre derivative ϑf is modular as well. In particular, the Serre derivatives of Eisenstein series appearing on the right of this equation are indeed modular forms. Moreover, the case distinction according to the sign of $l + 1 - k$, should be compared to the Taylor coefficients of the Jacobi form $G_1(z, w)$ in [24]. In fact, they are very similar (but here in level 1 and there in level 2, and here with Serre derivatives and there with usual derivatives) to the ones that appeared in [14] to prove the original assertion of Dijkgraaf from which the whole Bloch–Okounkov story arose.

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Data availability statement

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